

## Spacetime versus world-surface conformal invariance for particles, strings and membranes

E R C Abraham†, P S Howe‡§ and P K Townsend†

† Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, UK

‡ CERN, CH-1211 Geneva 23, Switzerland

Received 2 June 1989, in final form 4 July 1989

**Abstract.** The action for an extended object (string, membrane, etc) can be written in a form that is worldsheet or worldvolume (super)-Weyl invariant. We show that, as a consequence, such actions are never *spacetime* conformal invariant. In contrast, we also show that the action for a massless particle of spin 0,  $\frac{1}{2}$ , or 1, in an arbitrary spacetime background has symmetries corresponding to conformal isometries of the background, precisely because of the absence of worldline (super)-Weyl invariance.

Conformal invariance is one of those recurrent notions in elementary particle physics and general relativity. The equations of Maxwell, and Yang and Mills, are conformally invariant (for  $d = 4$  spacetime dimensions) and this fact is closely related to their renormalisability as quantum field theories. Einstein's general relativity is neither conformally invariant nor renormalisable, and various authors have suggested that the latter problem might be solved by replacing Einstein's theory at high energy by 'conformal gravity', for which the action is the square of the Weyl tensor. It must be admitted that the word conformal is being used here in two different senses; the conformal invariance of conformal gravity is actually a *gauge* invariance, often referred to as 'Weyl invariance', whereas the conformal invariance of Maxwell's equations is a *rigid* invariance with Noether charges that generate the Lie algebra of the conformal group  $SU(2, 2)$ . There is a sense in which conformal gravity is the gauge theory of  $SU(2, 2)$  [1]. One can also show that any Weyl-invariant theory of gravity plus (spin  $\leq 1$ ) matter will yield a theory with rigid  $SU(2, 2)$  invariance on restriction to flat spacetime or, more generally, a theory with symmetries corresponding to conformal isometries on restriction to some other background spacetime [2].

Conformal gravity has, therefore, a certain aesthetic appeal in that it promotes to a gauge invariance the maximal rigid symmetry of Maxwell's equations. It is worth remarking that renormalisability is not sufficient for the quantum consistency of a Weyl-invariant theory since *any* ultraviolet divergence will spoil Weyl invariance. A necessary condition for a sensible Weyl-invariant quantum field theory is finiteness. Remarkably, the only known Weyl-invariant quantum field theory is  $N = 4$  conformal supergravity [3, 4, 5], possibly coupled to  $N = 4$  super Yang-Mills theory. The Einstein-Hilbert action would have to appear as a result of a spontaneous breaking

§ Permanent Address: Mathematics Department, King's College, London, UK.

of conformal invariance [5, 6]. Potential difficulties with unitarity presumably account for the lack of interest in this *spacetime* conformally invariant approach to quantum gravity.

String theory is, in contrast, an approach to quantum gravity that accepts the existence of the Planck length as a fundamental unit (in the same sense that  $\hbar$  and  $c$  are fundamental units). String theory is therefore explicitly *not* spacetime conformally invariant. However, as is well known, conformal invariance or rather Weyl invariance still plays a role but now as a *worldsheet* symmetry. This is in fact true of all extended objects, not just strings. One standard form of the action for a  $p$ -dimensional extended object (a ' $p$ -brane'), in a background spacetime with metric  $g_{mn}(x)$  in local coordinates  $x^m$ , is

$$S_p = \int d^{p+1} \xi \det^{1/2} (\partial_i x^m \partial_j x^n g_{mn}(x)) \quad (1)$$

another equivalent form of the action introduces an independent auxiliary ( $p+1$ ) worldvolume metric  $\gamma_{ij}(\xi)$ ,

$$S_p = \frac{1}{2} \int d^{p+1} \xi \sqrt{-\gamma} [\gamma^{ij} \partial_i x^m \partial_j x^n g_{mn}(x) - (p-1)]. \quad (2)$$

This action is invariant under Weyl rescalings of  $\gamma_{ij}$  for  $p=1$  *only*. However there is another equivalent form of the action [7], also involving the auxiliary metric  $\gamma_{ij}$ , which is Weyl invariant for arbitrary  $p$

$$S_p = - \int d^{p+1} \xi \sqrt{-\gamma} (\gamma^{ij} \partial_i x^m \partial_j x^n g_{mn}(x))^{(p+1)/2}. \quad (3)$$

For  $p=0$  the actions (1) and (3) are identical, while for  $p=1$  the actions (2) and (3) are identical.

In the case of the particle,  $p=0$ , an action inequivalent to (3) can be obtained by omitting the  $(p-1)$  term in (2). This yields the action for a massless point particle,

$$S = \frac{1}{2} \int dt e^{-1} \dot{x}^m \dot{x}^n g_{mn}(x) \quad (4)$$

where  $e(t)$  is the worldline einbein. This action cannot be put into the manifestly Weyl-invariant form (3), and is clearly not invariant under a Weyl rescaling of  $e(t)$ . However, the action (4) does have *spacetime* conformal invariance. Given the general coordinate transformation  $x^m \rightarrow x^m + k^m(x)$ , where  $k^m(x)$  are the components of an infinitesimal vector field  $k$ , we find the variation

$$\delta S = \frac{1}{2} \int dt e^{-1} \dot{x}^m \dot{x}^n (\mathcal{L}_k g)_{mn} \quad (5)$$

where  $\mathcal{L}_k$  is the Lie derivative with respect to the vector field  $k$ . For a Killing vector  $\mathcal{L}_k g = 0$  and the action is invariant; i.e. isometries of spacetime yield symmetries of the particle action. This much is also true of extended objects; in fact the variation of the action (3) is

$$\delta S = - \int d^{p+1} \xi \sqrt{-\gamma} [(\partial x)^2]^{(p-1)/2} \gamma^{ij} \partial_i x^m \partial_j x^n (\mathcal{L}_k g)_{mn} \quad (6)$$

where  $(\partial x)^2 \equiv (\gamma^{ij} \partial_i x^m \partial_j x^n g_{mn})$ . The particle action (5), however, is invariant if  $k$  satisfies the weaker condition that it be a *conformal* Killing vector, i.e. if

$$(\mathcal{L}_k g)_{mn} = (2/d) g_{mn} k^r{}_{;r} \tag{7}$$

where  $d$  is the dimension of spacetime (and the semicolon indicates covariant differentiation with the Levi-Civita connection). This is because the resulting variation  $\delta S$  can be cancelled by a variation

$$\delta e = (2/d) k^r{}_{;r} \tag{8}$$

of the einbein. The wave equation in curved spacetime obtained by quantisation of the point particle is not uniquely determined by the classical action (4) because of operator ordering ambiguities. However, given a classical symmetry one can fix the factor ordering, at least partially, by requiring that the symmetry be respected by the quantisation procedure. For field theories this prescription can be spoiled by ‘anomalies’, but for the quantum-mechanical models considered here the only anomalies that can arise are global ones that affect only discrete symmetries [8]. Thus, by requiring that *all* the continuous symmetries of the classical action (4) be maintained by the quantum theory we obtain, upon quantisation, a conformally invariant wave equation.

The transformation (8) is a special case of a worldline Weyl transformation. Had the particle action been worldline Weyl invariant we would not have been able to use the variation (8) to cancel the  $k^r{}_{;r}$  terms in  $\delta S$ . That is, it is the *lack* of worldline Weyl invariance that allows the possibility of spacetime conformal invariance. One therefore suspects that the Weyl-invariant action (3) is not spacetime conformally invariant. Substituting (7) into (6) we find the variation

$$\delta S = -\frac{2}{d} \int d^{p+1} \xi \sqrt{-\gamma} [(\partial x)^2]^{(p+1)/2} k^r{}_{;r}. \tag{9}$$

It is easily seen that any candidate variation of  $\gamma_{ij}$  designed to cancel this must be proportional to  $\gamma_{ij}$ , and then it is obvious that, precisely because of the Weyl invariance of the action, a non-zero variation  $\delta S$  of (9) cannot be cancelled. This means, in turn, that the equations that result from quantisation of an extended object, e.g. string field equations, cannot be conformally invariant.

The above remarks illustrate the main point of this paper, which is that there is a kind of ‘exclusion principle’ between spacetime and worldsheet/worldvolume conformal invariance. In the remainder of this paper we report the results of a detailed check of this idea, principally in connection with an extension of the particle action (5) to one with  $N$ -extended worldline supersymmetry that describes a spin  $N/2$  point particle [9, 10].

In the presence of an arbitrary background spacetime the Hamiltonian form of the spinning particle action is [10]

$$S = \int dt \{ \dot{x}^m p_m + \frac{1}{2} i \lambda_i^a \lambda_i^b \eta_{ab} - \frac{1}{2} e [ g_{mn} (p_m - \frac{1}{2} i \lambda_i^a \lambda_i^b \omega_{mab}) (p_n - \frac{1}{2} i \lambda_j^c \lambda_j^d \omega_{ncd}) - \frac{1}{4} \lambda_i^a \lambda_i^b \lambda_j^c \lambda_j^d R_{abcd} ] - i \psi_i \lambda_i^a (p_m - \frac{1}{2} i \lambda_j^a \lambda_j^b \omega_{mab}) e_a^m - \frac{1}{2} i f_{ij} \lambda_i^a \lambda_i^b \eta_{ab} \} \tag{10}$$

where  $x^m, p_m$  are the coordinates of the  $2d$ -dimensional phase space of the particle,  $\lambda_i^m \equiv \lambda_i^a e_a^m(x)$ ,  $i = 1, 2, \dots, N$ , are the  $N$  worldline supersymmetry partners, with  $e_a^m$  the inverse of the spacetime vielbein  $e_m^a(x)$ ,  $g_{mn} \equiv e_m^a e_n^b \eta_{ab}$  the metric,  $\omega_{mab}$  the usual spin connection and  $R_{abcd}$  the corresponding curvature tensor. The Lagrange multiplier fields  $e$ ,  $\psi_i$  and  $f_{ij}$  are the worldline supergravity fields; the einbein, gravitini, and  $O(N)$  gauge fields, respectively. The  $N$ -extended supersymmetry transformations were given in [10] where it was shown that for  $N > 2$  the action is supersymmetric only if the spacetime background is flat.

It has also been shown [8, 11] that for a flat background the action has invariances corresponding to the conformal isometries of flat space. Here we shall extend this result to a curved spacetime background, in which case we are restricted by supersymmetry to  $N \leq 2$  for the reason just mentioned. We found that the action (10) is invariant under the infinitesimal transformations

$$\begin{aligned} \delta x^m &= k^m & \delta \lambda_i^a &= e^{am} e_b^n k_{[m;n]} \lambda_i^b - k^r \omega_r^a{}_b \lambda_i^b \\ \delta p_m &= -p_n \partial_m k^n - \frac{1}{2} i \lambda_i^a \lambda_i^b (e_a^r e_b^s k_{r;sm} - 2e_a^r e_c^s \omega_{mb}{}^c k_{[r;s]} - (k^r \omega_{rab})_{,m}) \end{aligned} \tag{11}$$

provided only that  $k$  is a conformal Killing vector satisfying (7), or, equivalently,

$$k_{(m;n)} = (1/d) g_{mn} k^r{}_{,r}. \tag{12}$$

This is because, as for the spinless particle, the resulting variation of  $\delta S$  is proportional to  $k^r{}_{,r}$  and can be cancelled by the following variation of the worldline supergravity fields,

$$\begin{aligned} \delta e &= (2/d) e k^r{}_{,r} \\ \delta \psi_i &= (1/d) \psi_i (k^r{}_{,r} - e e_a^n \lambda_i^a k^r{}_{,rn}) \\ \delta f_{ij} &= (2i/d) \psi_{[i} \lambda_j^a] e_a^m k^r{}_{,rm} - (ie/d) \lambda_i^a \lambda_j^b e_a^n e_b^s k^r{}_{,rsn}. \end{aligned} \tag{13}$$

The transformations (13) are in fact worldline super-Weyl transformations. This can most easily be seen from the superspace form of these results. The Lagrangian form of the  $N = 2$  superspace action was given in [8]. In the superconformal gauge it reads

$$S = -2 \int dt d^2\theta V^{-1} D\phi^m \bar{D}\phi^n g_{mn}(\phi) \tag{14}$$

where  $V(t, \theta, \bar{\theta})$  is the superfield containing the worldline supergravity fields, and  $\phi$  contains the dynamical fields  $x$  and  $\lambda$ . We refer to [8] for details of our superspace conventions. The transformations of (11) and (13), with the auxiliary momentum variable  $p_m$  eliminated, can now be written as

$$\delta \phi^m = k^m(\phi) \quad \delta V = (2/d) k^r{}_{,r}. \tag{15}$$

A general infinitesimal worldline super-Weyl transformation has the form  $V \rightarrow V + S$ , where  $S$  is an arbitrary worldline superfield. Thus  $\delta V$  in (15) is a special case of a super-Weyl transformation with  $S = (2/d) k^r{}_{,r}$ . It follows, by truncation, that the same is true for  $N = 1$ .

We have therefore verified that for  $N \leq 2$  the spinning particle action of [9, 10] has invariances corresponding to the conformal isometries of spacetime, by virtue of its *lack* of worldline (super)-Weyl invariance. For  $N > 2$  the same is true in a flat spacetime, as shown in [8, 11], but the conformal invariance of the field equations that result upon quantisation is specific to the *free* field equations and does not generalise when interactions with a gravitational background are included. That is, at present the only conformally invariant wave equations that we can obtain in curved spacetime from the quantisation of a point-particle model are those of spin 0,  $\frac{1}{2}$  or 1. It is likely that these are the only spins allowing consistent conformally invariant wave equations in curved space for  $d \geq 4$ .

One can similarly show that the spinning string action does *not* have invariances corresponding to the conformal isometries of spacetime (unless these happen to be ordinary isometries). This follows from the similar result for the bosonic string given earlier. For higher dimensional extended objects one might wonder whether anything is to be gained by using, for example, the form of the action (2) which is not world-volume super-Weyl invariant. We shall now investigate this point. The variation of (2) under the transformation  $x^m \rightarrow x^m + k^m$ ,  $\gamma^{ij} \rightarrow \gamma^{ij} + \delta\gamma^{ij}$ , with  $k^m$  the components of a conformal Killing vector but  $\delta\gamma^{ij}$  arbitrary, is

$$\delta S = \int d^{p+1}\xi \left\{ (1/d)k^r{}_{,r} \sqrt{-\gamma} (\partial x)^2 + \frac{1}{2} \delta\gamma^{ij} [\partial_i x^m \partial_j x^n g_{mn} - \frac{1}{2} \gamma_{ij} (\partial x)^2] \right. \\ \left. + \frac{1}{8} (p-1) \delta\gamma^{ij} \gamma_{ij} \right\}. \quad (16)$$

For  $p = 1$  the last term in (16) vanishes but the middle term cannot cancel the first because the coefficient of  $\delta\gamma^{ij}$  is traceless. For  $p \geq 2$  the last term in (16) must cancel independently of the other terms, and this requires  $\delta\gamma^{ij}$  to be traceless. However, if the first two terms in (16) are to cancel  $\delta\gamma^{ij}$  must be proportional to  $\gamma^{ij}$  and so no cancellation is possible.

This last result immediately extends to the case of spacetime supersymmetric extended objects, e.g. superstrings and supermembranes. The massless superparticle [12], however, must be investigated separately. In a general superspace background with superspace vielbein  $E_M^A = (E_M^a, E_M^\alpha)$  the action may be written as

$$S = \int d\tau e^{-1} \frac{1}{2} \dot{z}^N \dot{z}^M E_M^a E_N^b \eta_{ab} \quad (17)$$

where  $z^M = (x^m, \theta^{i\mu})$ ,  $i = 1, \dots, N$  are the coordinates of  $N$ -extended superspace. This action is obviously not worldline Weyl invariant and so, from our previous discussion, one would expect it to be spacetime superconformally invariant for those backgrounds admitting superconformal isometries. Superconformal transformations are defined to be superdiffeomorphisms which leave the supervielbein invariant up to super-Weyl transformations and local tangent space rotations. Super-Weyl transformations are local rescalings of the supervielbein with a (possibly complex) scalar superfield,  $W$ . In general this superfield will be constrained as these transformations must preserve the constraints which have been imposed on the superspace torsion tensor. However, it is always the case that  $E_M^a$  is scaled by a real (possibly constrained) scalar superfield,  $U$  ( $= W$ , when  $W$  is real). Thus for a superconformal Killing vector we have

$$(\mathcal{L}_k E)_M^a = U E_M^a + E_M^b L_b^a \quad (18)$$

where  $L_b^a$  is the parameter of a local Lorentz transformation in superspace, and where  $(\mathcal{L}_k E)_M^a \equiv \partial_M k^L E_L^a + k^L \partial_L E_M^a$  is the superspace Lie derivative of  $E_M^a$ . Equation (18) implies, as expected, that the component  $k^m|_{\theta=0}$  of a superconformal Killing vector  $k^M$  is an ordinary conformal Killing vector. For the case of flat  $d = 3, 4$  superspace it has previously been shown [13] that the action (17) is indeed superconformally invariant. It is easy to see that this extends to the other dimensions ( $d = 5, 6$ ) for which a flat superspace superconformal group exists, and in fact to any background admitting a superconformal Killing vector. Specifically, for  $k^M$  satisfying (18) the action (17) is invariant under the transformations  $\delta z^M = k^M$ ,  $\delta e = 2eU$ .

It is possible to construct spinning superparticle models [14, 15] with both worldline and spacetime supersymmetry. It would appear [15] that they are not superconformally invariant for  $d \geq 4$ . These models would therefore seem to deserve further investigation.

### Acknowledgments

One of the authors (EA) would like to thank the Commonwealth Scholarship Commission for their financial support.

### References

- [1] Kaku M, Townsend P K and van Nieuwenhuizen P 1977 *Phys. Lett.* **69B** 304
- [2] Zumino B 1970 *Lectures in Elementary Particle Physics and Quantum Field Theory (Proc. 1970 Brandeis Summer Institute)* vol 2 (Cambridge, MA: MIT Press)
- [3] Bergshoeff E, de Roo M and de Wit B 1981 *Nucl. Phys. B* **182** 173  
Howe P S 1981 *Phys. Lett.* **100B** 389
- [4] Howe P S, Stelle K S and Townsend P K 1984 *Nucl. Phys. B* **236** 125
- [5] Fradkin E S and Tseytlin A A 1985 *Phys. Rep.* **119** 233
- [6] Kaku M and Townsend P K 1978 *Phys. Lett.* **76B** 54  
Adler S 1982 *Rev. Mod. Phys.* **54** 729
- [7] Lindström U and Roček M 1989 *Phys. Lett.* **217B** 208  
Alves M S and Barcelos-Neto J 1988 *Europhys. Lett.* **7** 395
- [8] Howe P S, Penati S, Pernici S and Townsend P K 1989 *Class. Quantum Grav.* **6** 1125–39
- [9] Gershun V D and Tkach V I 1979 *JETP Lett.* **29** 320
- [10] Howe P S, Penati S, Pernici S and Townsend P K 1988 *Phys. Lett.* **215B** 555
- [11] Siegel W 1988 *Int. J. Mod. Phys. A* **3** 2713; 1988 All free conformal representations in all dimensions  
*Stony Brook Preprint* no ITP-SB-88-45
- [12] Brink L and Schwarz J H 1981 *Nucl. Phys. B* **185** 221
- [13] Schwarz J H 1981 *Phys. Lett.* **100B** 310
- [14] Kowalski-Glikman J 1988 *Phys. Lett.* **202B** 343  
Aoyama S, Kowalski-Glikman J, van Holten J W and Lukierski J 1988 *Phys. Lett.* **201B** 487; 1989  
*Phys. Lett.* **216B** 133  
Gates S J and Nishino H 1986 *Class. Quantum Grav.* **3** 391  
Brooks R, Muhammad F and Gates S J 1986 *Class. Quantum Grav.* **3** 745
- [15] Bergshoeff E and van Holten J W 1989 The spectrum of spinning superparticles *Preprint* no CERN-TH.5383/89