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INTERSECTING EXTENDED OBJECTS IN SUPERSYMMETRIC FIELD THEORIES

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We show that there are three cases for which the generic intersections of p-dimensional extended object solutions of a supersymmetric field theory in a d-dimensional space-time are stringlike. They are (i) $d=4$, $p=2$, (ii) $d=6$, $p=3$ and (iii) $d=10$, $p=5$. By consideration of the topological charges associated with these objects we obtain a necessary condition for stable stringlike intersections to occur, a condition that is satisfied by the first two cases but *not* the third, which may have implications for the "heterotic 5-branes" recently discussed by Strominger. For case (i) we show by an analysis of the $d=4$ Wess–Zumino model that stringlike intersections of domain walls can indeed occur.

1. Introduction

Relativistic field theories often allow the formation of topological defects which, at low energies, appear to be extended objects such as strings or membranes. The dynamics of such objects is a well studied subject. Less well studied is the issue of whether such extended objects can intersect and if so what governs the dynamics of the intersection. In a d-dimensional space-time, p-dimensional extended objects, with $(p+1)$-dimensional worldvolumes, will generically intersect in such a way that the intersection is an extended object of dimension

$$p_{\text{int}} = p - (d - p - 1)\,. \tag{1.1}$$

For supersymmetric extended objects the possible values of p and d are severely limited, there being four families labelled $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, for which the worldvolume codimension $(d-p-1)$ equals $1,2,4,8$ respectively [1]. For each family there is a maximum value $p_{\max}$ of p, equal to $2,3,5,2$ respectively from which it is clear that $p_{\text{int}} \leqslant 1$, with equality only for $p = p_{\max}$ and then only for the $\mathbb{R}, \mathbb{C}, \mathbb{H}$ families. There are therefore only three instances for which the generic intersections of an extended object of a supersymmetric field theory will themselves be extended.

They are

$$\begin{array}{lll} \text{(i)} & d=4 & p=2 \\ \text{(ii)} & d=6 & p=3 \\ \text{(iii)} & d=10 & p=5 \end{array}$$

and in each case $p_{\mathrm{int}} = 1$.

Examples of supersymmetric field theories in dimensions $d = 4, 6, 10$ allowing topological defects of dimension $p = 2, 3, 5$, respectively, are known. In particular, it was pointed out by one of the authors [2] that the usual $d = 4$ Yang–Mills (YM) instanton is an example of a super 5-brane when viewed as a solution of $d = 10$ super YM theory. This solution was recently extended by Strominger [3] to include the gravitational and other fields of the field theory limit of the $d = 10$ heterotic string, and it was suggested that the dynamics of the stringlike intersections would be governed by the Green–Schwarz superstring action.

Teitelboim has shown that there is a duality of electric/magnetic type between extended objects of dimensions p and $d - p + 4$ [4]; this is closely related to the duality between field strengths of antisymmetric tensor gauge potentials of ranks $p + 1$ and $d - p - 3$. Therefore, given an extended object of dimension p, the dual extended object, in the sense of [4], has dimension

$$p_{\mathrm{dual}} = d - p - 4\,. \tag{1.2}$$

Comparing with (1.1) we see that $p_{\mathrm{dual}} = p_{\mathrm{int}}$ requires $p = \frac{1}{3}(2d - 5)$, in which case $p_{\mathrm{int}} = p_{\mathrm{dual}} = \frac{1}{3}(d - 7)$. Then $p_{\mathrm{int}} = 1$ implies $d = 10$; i.e. case (iii) above, the $d = 10$ 5-brane, is the only one of the three for which there is a duality between the extended object and its stringlike intersections.

It was emphasised by Duff [5] that the existence of two dual forms of $d = 10$, $N = 1$ supergravity (with either a second-rank or sixth-rank antisymmetric tensor gauge potential) is possibly a reflection of a similar duality between the $d = 10$ superstring and super 5-brane. This dual relationship fails for cases (i) and (ii) however; e.g. for $d = 6$, $p = 3$ we do *not* find that $p_{\mathrm{dual}} = 1$, but we do find that $p_{\mathrm{int}} = 1$, so it is the notion of intersections, rather than field strength duality, that unifies the four families of extended objects. It happens that both notions agree for $d = 10$, a fact that was exploited in ref. [3].

Despite this special feature of the $d = 10$ 5-brane, it is still of interest to consider cases (i), (ii) and (iii) together when discussing whether stringlike intersections of supersymmetric extended objects will in fact occur. We shall show that there is a necessary condition for the stability of any generic intersection (necessary at least for the type of extended object solution we envisage here) which is satisfied by cases (i) and (ii) but *not* by case (iii).

The point at issue is as follows. We imagine that the cores of two p-dimensional objects (or possibly distant parts of one such object) overlap at rest, at least across a p-dimensional region that is large compared to the size of the objects' cores. If this happens it may be possible for the two cores to fuse, thereby forming the core of a third p-dimensional object. This would result in the formation of intersections of three p-branes. However, this can happen only if the energy per unit p-volume, i.e. the tension, of the newly formed p-brane is less than the sum of the tensions of the original p-branes, and then only if the third type of p-brane actually exists. It might be supposed that these conditions are easily met by postulating a single type of p-brane with tension M, as $M < 2M$, but this overlooks the possibility that the process might violate a conservation law. In fact, for all known extended object solutions of supersymmetric field theories there *is* a conservation law because the tension M is the magnitude of an additive topological charge carried by the object. For a p-dimensional object this charge is a pth rank antisymmetric tensor, but it is clear that in a small region of overlap it is a good approximation to ignore dependence on the coordinates of the p directions defining the p-plane of the objects. We can therefore analyse the process in terms of particle-like solutions, which we shall refer to as "solitons", of a $(d-p)$-dimensional supersymmetric field theory obtained by dimensional reduction. In the process of reduction the pth rank charge becomes an ordinary Lorentz scalar charge. If two solitons of topological charge T fuse they must, by charge conservation, form a third soliton of topological charge $2T$. The crucial point is that the topological charge T of a soliton appears in the supersymmetry algebra as a central charge [6] and that, as a consequence, one can prove that

$$M \geqslant |T|, \tag{1.3}$$

where M is now the soliton's mass. The mass of the third soliton is therefore at least $|2T|$. In practice one finds that the bound (1.3) is always saturated, so that the process of soliton fusion is energetically possible but has zero phase space.

If this were all, we could immediately conclude that supersymmetric extended objects can never fuse to form stable intersections (stable in the sense that energy is required to remove them) but there is an obvious way in which the above reasoning might be circumvented. Returning to the soliton analysis, the topological charge, although a Lorentz scalar, may be a vector in some internal space. Addition of topological charges is then vector addition. For example, suppose that the topological charge T is a complex number. Then the mass M_3 of a soliton formed by fusion of two other solitons of masses M_1 and M_2, with complex topological charges T_1 and T_2, respectively, is

$$M_3 = |T_1 + T_2| \leqslant |T_1| + |T_2| = M_1 + M_2. \tag{1.4}$$

The inequality is saturated if and only if the phases of T_1 and T_2 are equal. Otherwise, $M_3 < M_1 + M_2$ and the fusion of the two solitons is an exothermic reaction, i.e. the third soliton will be stable. However, given the existence of this third soliton, the stability of the other two requires that $M_1 < M_2 + M_3$ and $M_2 < M_3 + M_1$. Thus the phases of all three topological charges T_1, T_2 and T_3 must differ. This cannot be satisfied for real charges (for which at most two are of opposite sign).

We can now translate this discussion about solitons of a $(d-p)$-dimensional field theory into one about p-brane solutions of the original d-dimensional field theory. The central charge in the $(d-p)$-dimensional supersymmetry algebra becomes a pth rank charge in the d-dimensional algebra. This can be demonstrated either directly from the field theory [7] or in a model-independent way from the super p-brane action [8]. We are therefore led to the following criterion: a necessary condition for the formation of a stable stringlike intersection of p-dimensional objects of a supersymmetric field theory is that the pth rank charge in the supersymmetry algebra must be a vector in some internal space of dimension $\geqslant 2$. This does not mean that we could not set up an initial configuration for which a collection of p-dimensional objects would *appear* to have stringlike intersections but, if our criterion is not satisfied, these will disappear because their stability is, at best, marginal. We have discussed only intersections at which three p-branes meet but the result is easily extended to intersections of higher degree.

In sect. 2 we shall show that our criterion for the existence of stringlike intersections is satisfied for cases (i) and (ii) but *not* for case (iii). In the latter case the analysis may be reduced, as explained above, to one involving solitons in a $d = 5$, $N = 4$ supersymmetric field theory. A suitable model is $N = 4$ super YM theory with the YM instanton as the soliton core. The number of possible central charges in this theory is 6. They form a carrier space for the $\mathbf{5} \oplus \mathbf{1}$ representation of the supersymmetry automorphism group USp(4). If the soliton has a topological charge that is a vector in the 5-dimensional space then any two of them need not have parallel 5-vector charges. Therefore, in principle, two solitons of $N = 4$, $d = 5$ super YM theory could fuse to form a third. However, one must be careful in drawing conclusions about $d = 10$ super YM theory from this fact. The point is that the 5-vector charges do not originate from the topological 5-index charge of the $d = 10$ supersymmetry algebra. They have a standard Kaluza–Klein (KK) origin as components of the momentum in the compactified directions. For the standard 5-brane solution of $d = 10$ super YM theory these charges vanish.

Because of the importance of correctly identifying the higher-dimensional origin of the central charges of supersymmetry algebras we present in sect. 2 a complete analysis of this question. Briefly, all central charges not of standard KK type originate from antisymmetric tensor charges in dimension 4, 6, 10 or 11.

Having determined that for cases (i) and (ii) the intersection of membranes or 3-branes, respectively, is energetically possible, it remains to show that there exist models for which it happens. In this paper we shall discuss in detail only case (i) for which a suitable model is the $d = 4$ Wess–Zumino (WZ) model with polynomial holomorphic superpotential W. This model admits domain walls, for which the effective action is the $d = 4$ supermembrane action, as discussed in ref. [9], following ref. [10]. Having shown in sect. 2 that there is a phase freedom for the two-index topological charge in $d = 4$, we carry through the analysis in sect. 3 in terms of the dimensionally reduced $d = 2$ theory, for which the solitons carry the complex topological charge

$$T = 2\int_{-\infty}^{+\infty} \mathrm{d}\sigma \frac{\partial}{\partial\sigma} W. \tag{1.5}$$

We show that there exist superpotentials W such that two solitons can fuse to form a third. It turns out that the condition on W for this to be possible is not simply that there exist solitons carrying topological charges of differing phases; this is necessary but not sufficient. We give a complete analysis for quartic W.

2. The higher-dimensional origin of central charges

Extended super-Poincaré algebras, in whatever dimension d of space-time, generally allow central charges, i.e. charges that commute with all those of the super-Poincaré algebra (and with each other). In many cases these central charges have a Kaluza–Klein (KK) origin in that they can be understood as the (quantized) values of the momenta in the "extra" periodically identified directions. A much studied example is $N = 2$, $d = 4$ for which two real central charges are possible; this may be seen to be a consequence of the fact that $d = 4$ field theories with $N = 2$ supersymmetry can generally be obtained by the dimensional reduction of a $d = 6$ supersymmetric field theory. Similarly, a $d = 4$ field theory with $N = 4$ supersymmetry can generally be obtained by dimensional reduction from $d = 10$. This might lead one to suspect that the $N = 4$ supersymmetry algebra should allow a total of $10 - 4 = 6$ central charges whereas, in fact, it allows 12 [11]. What is the origin of the other 6? In this section we find the higher-dimensional origin of *all* central charges of all supersymmetry algebras for $1 \leqslant d \leqslant 11$ and such that the total number of real components of the supersymmetry charges does not exceed 32. These conditions are satisfied by all interacting supersymmetric field theories. As might be expected, any "additional" central charges, i.e. those not interpretable as values of momenta in the compactified dimensions, arise from *non-central* charges of the higher-dimensional super Pointcaré algebra. By "non-central" we mean charges that fail to be central only because they carry a non-trivial representation of the Lorentz group. Van Holten and Van Proeyen have made a thorough analysis of which non-central charges can in principle occur [12]. They included spinorial charges which may appear in the commutator of P_m (space-time translation

TABLE 1
Number of central charges of d-dimensional supersymmetry algebras

Dimension	Number of central charges	Total
1	$\frac{1}{2}N(N+1)-1$	N
2	$N_L - N_R$	$N_L + N_R$
3	$\frac{1}{2}N(N-1)$	$2N$
4	$N(N-1)$	$4N$
5	$\frac{1}{2}N(N-1)$ (N even)	$4N$
6	$N_L N_R$	$8N_L + 8N_R$
7	$\frac{1}{2}N(N+1)$ (N even)	$8N$
8	$N(N+1)$	$16N$
9	$\frac{1}{2}N(N+1)$	$16N$
10	$N_L N_R$	$16N_L + 16N_R$

The number in the last column is the total number of (real) components of the supersymmetry charges.

generator) with the Q^i_α (supersymmetry generators), a possibility that has been recently exploited for other purposes [13]. Here we shall be interested only in the tensorial charges $T_{m_1 \ldots m_k}$, antisymmetric in all k indices, which may appear in the anticommutator of two supersymmetry charges.

Our main result is simply stated. Firstly, we remark that if a d-dimensional supersymmetric field theory can be obtained by dimensional reduction from a field theory in dimension $D > d$ with the minimal number of supersymmetries for that dimension, then $D = 3$, 4, 6, 10 or 11. The "surplus" central charges in the d-dimensional supersymmetry algebra, i.e. other than the $D - d$ charges of conventional KK origin, can then be seen to arise from the following non-central charges in dimensions $D = 4$, 6, 10 or 11:

$$D = 4 \qquad T^{+}_{mn}$$

$$D = 6 \qquad T^{+(ij)}_{mnp}$$

$$D = 10 \qquad T^{+}_{mnpqr}$$

$$D = 11 \qquad T_{mn}, \quad T_{mnpqr}.$$

Here $+$ indicates selfdual and all charges are real except for the $D = 4$ charge T^{+}_{mn}, which is complex (its complex conjugate being anti-selfdual). The ij indices for $D = 6$ indicate that the charge is an SU(2) triplet ($i, j = 1,2$). Observe that there are no possible *central* charges in the minimal supersymmetry algebra in these dimensions and that the given tensorial charges are the only ones allowed by the Jacobi identities [12].

We shall need to know how many central charges are possible for d-dimensional extended super-Poincaré algebras. The numbers are easily found from a case by case inspection for the relevant dimensions, i.e. $1 \leqslant d \leqslant 10$. The result is given in table 1 as a function of the number N of supersymmetries, except for $d = 2, 6, 10$ for which it is given for (N_L, N_R)-supersymmetry as a function of N_L and N_R, the number of chiral and anti-chiral supersymmetry charges. Note that for $d = 5$ and 7 the minimal number of supersymmetries is, by convention, $N = 2$ rather than $N = 1$, because the minimal supersymmetry algebra in these dimensions has an SU(2) automorphism group.

Let n equal the total number of (real) components of the supersymmetry generators (in parentheses in table 1). If a d-dimensional algebra is non-chiral and has n equal to 2, 4, 8, 16, or 32 it can be viewed as the dimensional reduction of a minimal supersymmetry algebra in the higher dimension $D = 3$, 4, 6, 10 or 11, respectively. Clearly not all supersymmetry algebras in $d \leqslant 10$ satisfy these conditions (e.g. $(1,2)$ supersymmetry for $d = 2$ and $N = 3$ for $d = 4$) but those that do not can be obtained by truncation of one that does (e.g. $(2,2)$ in $d = 2$ and $N = 4$ in $d = 4$). We may therefore restrict ourselves to those cases for which these conditions are satisfied.

As an example consider $d = 5$, $N = 4$. Since $n = 16$ we can start in $D = 10$ where we have the tensorial charge T^+_{mnpqr}. On dimensional reduction from $D = 10$ to $d = 5$ this yields the real central charge

$$T = T^+_{56789}\,. \tag{2.1}$$

There are also five real central charges of obvious KK origin, making a total of $5 + 1 = 6$ possible central charges. We can see from the table that this is precisely the right number. As another example consider $d = 3$, $N = 4$. Since $n = 8$ we can start in $D = 6$ where we have the SU(2) triplet of tensorial charges $T^{+(ij)}_{mnp}$. On dimensional reduction from $D = 6$ to $d = 3$ this yields the triplet of central charges

$$T^{(ij)} = T^{+(ij)}_{345}\,. \tag{2.2}$$

Together with the three central charges of KK origin, this makes a total of six possible central charges. Again, we see from table 1 that this is the right number. Observe that in both examples the central charges coming from the higher-dimensional tensorial charges and those of KK origin belong separately to representations of the automorphism group of the lower-dimensional algebra (U Sp(4) in the first case and SO(3) × SO(3) in the second).

Another example is $d = 2$, $p = q = 2$. Since $n = 4$ we can start in $D = 4$. The *complex* selfdual antisymmetric tensor charge T^+_{mn} now gives rise to a *complex* central charge in $d = 2$,

$$T = T^+_{23}\,. \tag{2.3}$$

Another way of putting this is that given the *real* antisymmetric tensor charge T_{mn} both it and its dual contribute to the central charges in lower dimensions. This must be borne in mind when reducing from $D=11$. For example the $d=5$, $N=8$ supersymmetry algebra has a total of 28 central charges of which 6 are of KK origin. Of the remaining 22, 15 come from T_{mn}, 6 from T_{mnpqr} and 1 from the dual of T_{mnpqr} (i.e. $T_{01234}=\tilde{T}_{5678910}$). In this case the central charges of different origin must be combined to form a representation of the lower dimensional automorphism group U Sp(8).

We come now to the question of whether the tensorial non-central charges in dimensions 4, 6, 10, and 11 have a physical realisation. It was pointed out by Zizzi [7] that for super Yang–Mills theory in a flat space-time of topology $\mathbb{R}^4 \times \mathrm{T}^5$ the $d=4$ YM instanton is a configuration for which the fifth-rank charge T^{+}_{mnpqr} appears in the ten-dimensional supersymmetry algebra as a topological charge, and in ref. [2] that such a configuration could be interpreted as a five-dimensional extended object for which the effective action is that of the $d=10$ super 5-brane. It was further shown in ref. [8] that, owing to the presence of a Wess–Zumino term in this action, the supersymmetry algebra is modified to include a fifth-rank charge T^{+}_{mnpqr}. The advantage of the effective action derivation is that it is model independent. We should be able to deduce the presence of the charges T^{+}_{mn} and $T^{+(ij)}_{mnp}$ in a similar fashion from the actions for the four-dimensional supermembrane and the six-dimensional super 3-brane, respectively. In fact, this already follows from the general analysis of ref. [8] except that the freedom of choice for the Wess–Zumino term in the action, and hence of the topological charge in the algebra, was not taken into account. This gap is easily filled.

The Wess–Zumino term in the $d=4$ supermembrane action is constructed from the exact super-Poincaré invariant superspace 4-form $H=\Pi^m\Pi^n\,\mathrm{d}\bar{\theta}\,\Gamma_{mn}\,\mathrm{d}\theta$, where $\Pi^m=\mathrm{d}x^m-i\bar{\theta}\Gamma^m\,\mathrm{d}\theta$. If we now make the substitution $\theta\to \mathrm{e}^{\gamma_5\phi/2}\theta$, with ϕ a constant phase, we find that

$$H\to \Pi^m\Pi^n\,\mathrm{d}\bar{\theta}\,\Gamma_{mn}\,\mathrm{e}^{\gamma_5\phi}\,\mathrm{d}\theta\,, \tag{2.4}$$

whereas all other terms in the supermembrane action are unaffected. There is therefore a phase freedom in the topological charge of the four-dimensional supermembrane.

The WZ term of the $d=6$ super 3-brane action is related to an exact 5-form. In the $d=6$ symplectic-Majorana spinor notation [14] this 5-form is

$$H=\Pi_{\alpha\beta}\Pi^{\beta\gamma}\Pi_{\gamma\delta}\,\mathrm{d}\theta^{\alpha i}\,\mathrm{d}\theta^{\delta j}\,\xi_{ij}\,, \tag{2.5}$$

where the constants ξ_{ij} must satisfy $\frac{1}{2}\xi_{ij}\xi^{ij}=1$ (summed over i,j). The action given in ref. [10] corresponds to the choice $\xi_{ij}=\delta_{ij}$ but by an SU(2) rotation we can achieve any other allowed choice for H. All other terms in the action are constructed from the components of the SU(2)-invariant form $\Pi^{\alpha\beta} = \mathrm{d}X^{\alpha\beta} - i\theta^{\alpha i}\,\mathrm{d}\theta^{\beta j}\,\epsilon_{ij}$. It follows that there is a three-dimensional freedom in the topological charge of the $d=3$ super 3-brane.

Note that the crucial point in the above analysis is that the supersymmetry automorphism group is non-trivial, U(1) for $d=4$ and SU(2) for $d=6$. For $d=10$ (and $N_{\mathrm{L}}=1$, $N_{\mathrm{R}}=0$) this group is trivial so there is no internal degree of freedom for the 5-index charge. As explained in sect. 1, this fact means that stringlike intersections of 5-branes are not stable.

Concerning the physical realisation of the $D=11$ charges T_{mn} and T_{mnpqr}, we remark that, although there is no known domain wall solution of 11-dimensional supergravity, there is an 11-dimensional supermembrane action [15] for which a charge T_{mn} appears in the supersymmetry algebra. We further remark that, although there is no known 11-dimensional super 5-brane (which might be associated with T_{mnpqr}), a 5-brane in $d=11$ is dual, in the sense of ref. [4], to a membrane.

We now proceed to give a more detailed analysis of case (i), intersecting supermembranes in $d=4$.

3. Supermembranes and superparticles as effective actions

We have seen that, for our purposes, the study of intersecting domain walls in a (3 + 1)-dimensional space-time can be reduced to a study of solitons in a (1 + 1)-dimensional space-time. As the effective action for a single domain wall is that of the $d=4$ supermembrane the corresponding effective action for the soliton will be that of the (2,2) supersymmetric $d=2$ superparticle, obtained by dimensional reduction. This action is

$$S = -M\int \mathrm{d}t\,\sqrt{(\omega^{\neq}\omega^{=})} + \left[\frac{i}{2}T\int \mathrm{d}t\,\theta^{+}\dot{\theta}^{-} + \text{c.c.}\right], \tag{3.1}$$

where M is the particle's mass, T is a complex number with dimensions of M, and

$$\omega^{\neq} = \dot{x}^{\neq} - i\bar{\theta}^{+}\dot{\theta}^{+} - i\theta^{+}\dot{\bar{\theta}}^{+}, \qquad \omega^{=} = \dot{x}^{=} - i\bar{\theta}^{-}\dot{\theta}^{-} - i\theta^{-}\dot{\bar{\theta}}^{-}. \tag{3.2}$$

Here θ^{+}, θ^{-} are a pair of complex (anti)chiral anticommuting spinors of SO(1,1), and the number of + or − indices indicates the SO(1,1) charge. The first term in (3.1) is manifestly invariant under the (2,2) space-time supersymmetry transformations

$$\delta_\epsilon\theta^{+} = \epsilon^{+}, \qquad \delta_\epsilon\theta^{-} = \epsilon^{-}, \qquad \delta_\epsilon x^{\neq} = i\epsilon^{+}\theta^{+} + i\bar{\epsilon}^{+}\bar{\theta}^{+}, \qquad \delta_\epsilon x^{=} = i\epsilon^{-}\theta^{-} + i\bar{\epsilon}^{-}\bar{\theta}^{-}. \tag{3.3}$$

The second term in (3.1) is a WZ term for the supertranslation group and is not manifestly invariant. As a consequence the supersymmetry algebra is modified to include T as a central charge, as follows:

$$\{Q_+,\bar{Q}_+\}=P_{\neq}\,,\qquad \{Q_-,\bar{Q}_-\}=P_{=}\,,$$
$$\{Q_+,\bar{Q}_-\}=0,\qquad \{Q_+,Q_-\}=T\,, \tag{3.4}$$

where $P_{\neq}$ and $P_{=}$ are the self-dual and anti-selfdual components of the two-momentum. From the supersymmetry algebra we can derive a lower bound on the particle's mass M. Consider the hermitian charge

$$Q(\alpha)=e^{-i\alpha/2}(Q_++Q_-)+e^{i\alpha/2}(\bar{Q}_++\bar{Q}_-), \tag{3.5}$$

for which

$$Q(\alpha)^2=P_{\neq}+P_{=}+2\,\mathrm{Re}(e^{-i\alpha}T)\geqslant 0. \tag{3.6}$$

Since $P_{\neq}+P_{=}=2P_0=-2H$, where H is the hamiltonian, it follows, by choosing $\alpha=\arg T$, that $H\geqslant |T|$. For a particle at rest $H=M$ (its mass); we therefore have the bound

$$M\geqslant |T|\,. \tag{3.7}$$

When this bound is saturated the action acquires a fermionic gauge invariance with transformations

$$\delta_\kappa x^{\neq}=\tfrac{1}{2}i\delta_\kappa\theta^+\bar{\theta}^++\text{c.c.},\qquad \delta_\kappa x^{=}=\tfrac{1}{2}i\delta_\kappa\theta^-\bar{\theta}^-+\text{c.c}.$$
$$\delta_\kappa\theta^+=-\frac{M}{2}\sqrt{\frac{\omega^{\neq}}{\omega^{=}}}\,\kappa_+-\tfrac{1}{2}\bar{T}\bar{\kappa}_-\,,\qquad \delta_\kappa\theta^-=-\frac{M}{2}\sqrt{\frac{\omega^{=}}{\omega^{\neq}}}\,\kappa_--\tfrac{1}{2}\bar{T}\bar{\kappa}_+\,. \tag{3.8}$$

This "κ-symmetry" allows one complex combination of θ^+ and θ^- to be gauged away, after which the action depends on only a single complex variable.

The above analysis is model independent. A suitable model is provided by the $d=4$ WZ model with holomorphic superpotential W. Upon dimensional reduction to $d=2$ the action becomes

$$S=\int d^2\sigma\Big[\partial_{=}z\partial_{\neq}\bar{z}+i\bar{\psi}_+\partial_{=}\psi_++i\bar{\psi}_-\partial_{\neq}\psi_-$$
$$-iW''(z)\psi_+\psi_--i\overline{W''(z)}\,\bar{\psi}_+\bar{\psi}_--|W'(z)|^2\Big]\,, \tag{3.9}$$

where $\boldsymbol{\sigma}=(t,\sigma)$ are the coordinates of $(1+1)$-dimensional Minkowski space, $z(\boldsymbol{\sigma})$

is a complex scalar field and $\psi_+(\sigma), \psi_-(\sigma)$, are complex one-component spinor fields of chirality $+1, -1$, respectively, $W(z)$ is the holomorphic superpotential (with $W' = \mathrm{d}W/\mathrm{d}z$), and

$$\partial_{\neq} = \partial_0 + \partial_1, \qquad \partial_{=} = \partial_0 - \partial_1 . \tag{3.10}$$

The action (3.9) is invariant under the following supersymmetry transformations with complex infinitesimal anticommuting parameters ϵ_+, ϵ_-:

$$\delta z = i\epsilon_+ \psi_- + i\epsilon_- \psi_+ , \tag{3.11a}$$

$$\delta\psi_+ = -\partial_{\neq} z \bar{\epsilon}_- - \overline{W'(z)}\, \epsilon_+ ,$$

$$\delta\psi_- = -\partial_{=} z \bar{\epsilon}_+ + \overline{W'(z)}\, \epsilon_- . \tag{3.11b}$$

The corresponding complex chiral supersymmetry charges Q_+ and Q_- are

$$Q_+ = \int \mathrm{d}\sigma \big[\partial_{\neq} z \bar{\psi}_+ - W'(z) \psi_- \big] ,$$

$$Q_- = \int \mathrm{d}\sigma \big[\partial_{=} z \bar{\psi}_- + W'(z) \psi_+ \big] . \tag{3.12}$$

Replacing $\partial_0 z$ by p, and using the (anti)commutation relations

$$[p, \bar{z}] = -i, \qquad \{\psi_+, \bar{\psi}_+\} = \{\psi_-, \bar{\psi}_-\} = 1 \tag{3.13}$$

and their complex conjugates (all others vanishing), we recover the supersymmetry algebra (3.4) with the topological charge T being given by

$$T = 2\int_{-\infty}^{+\infty} \mathrm{d}\sigma \frac{\partial}{\partial\sigma} W . \tag{3.14}$$

This generalises the result of Witten and Olive [6] to extended supersymmetry.

For time-independent field configurations the equations of motion that follow from (3.9) reduce to

$$\partial_\sigma^2 z - W'(z)\overline{W''(z)} - i\overline{W'''(z)}\, \bar{\psi}_+ \bar{\psi}_- = 0 , \tag{3.15a}$$

$$\begin{pmatrix} \partial_\sigma & \overline{W''(z)} \\ W''(z) & \partial_\sigma \end{pmatrix} \begin{pmatrix} \psi_+ \\ \bar{\psi}_- \end{pmatrix} = 0 . \tag{3.15b}$$

Given a soliton solution for $z(\sigma)$ of (3.15a), with $\psi_+ = \psi_- = 0$, its mass is

$$M = \int_{-\infty}^{\infty} \mathrm{d}\sigma \left[|\partial_\sigma z|^2 + |W'(z)|^2 \right]$$
$$= \int_{-\infty}^{\infty} \mathrm{d}\sigma \left| \partial_\sigma z - \mathrm{e}^{i\alpha} \overline{W'(z)} \right|^2 + \mathrm{Re}(\mathrm{e}^{-i\alpha} T) \tag{3.16}$$

for arbitrary phase α. Since α is arbitrary we obtain the strongest lower bound on M by choosing $\alpha = \arg T$. Then

$$M \geqslant |T| , \tag{3.17}$$

as we found previously from our analysis of the supersymmetry algebra. The bound is saturated by solutions of the first-order equation

$$\partial_\sigma z - \mathrm{e}^{i\alpha} \overline{W'(z)} = 0 \tag{3.18}$$

with $\alpha = \arg T$, which are easily seen to be solutions of the second-order equation (3.15a) when $\psi_+ = \psi_- = 0$. Multiplying (3.18) by $W'(z)$ and integrating over σ, we find that

$$T = \mathrm{e}^{i\alpha} 2 \int_{-\infty}^{\infty} \mathrm{d}\sigma |W'(z)|^2 , \tag{3.19}$$

so that α indeed equals $\arg T$. Clearly, the only solutions of (3.18) with finite T (and hence finite mass) are those which interpolate between two critical points of $W(z)$.

A feature of soliton configurations for which the bound (3.17) is saturated is that only half, rather than all, of the supersymmetry is broken [6]. To see this we define $\eta = \frac{1}{2}(\epsilon_+ - \epsilon_-)$ and $\zeta = -(\epsilon_+ + \epsilon_-)$ and rewrite (3.11b), for $\dot{z} = 0$, as

$$\begin{pmatrix} \delta\psi_+ \\ \delta\bar{\psi}_- \end{pmatrix} = \begin{pmatrix} \left(\partial_\sigma z - \mathrm{e}^{i\alpha} \overline{W'(z)}\right) & \left(\partial_\sigma z + \mathrm{e}^{i\alpha} \overline{W'(z)}\right) \\ \mathrm{e}^{i\alpha}\left(\partial_\sigma \bar{z} - \mathrm{e}^{-i\alpha} W'(z)\right) & -\mathrm{e}^{i\alpha}\left(\partial_\sigma \bar{z} + \mathrm{e}^{-i\alpha} W'(z)\right) \end{pmatrix} \begin{pmatrix} \eta \\ \zeta \end{pmatrix} . \tag{3.20}$$

Configurations satisfying eq. (3.18) therefore have the property that they are preserved by supersymmetry transformations with $\zeta = 0$, $\eta \neq 0$, i.e. $\epsilon_+ + \epsilon_- = 0$, whereas they are not preserved by those with $\eta = 0$, $\zeta \neq 0$. This has the following consequence: for any solution of eq. (3.15a) with $\psi_+ = \psi_- = 0$ a solution of (3.15b) is given by

$$\begin{pmatrix} \psi_+ \\ \bar{\psi}_- \end{pmatrix} = \begin{pmatrix} \left(\partial_\sigma z - e^{i\alpha}\overline{W'(z)}\right)\eta + \left(\partial_\sigma z + e^{i\alpha}\overline{W'(z)}\right)\zeta \\ e^{i\alpha}\left[\left(\partial_\sigma \bar{z} - e^{-i\alpha}W'(z)\right)\eta - \left(\partial_\sigma \bar{z} + e^{-i\alpha}W'(z)\right)\zeta\right] \end{pmatrix} \tag{3.21}$$

for any complex constants η, ζ. When $\partial_\sigma z = e^{i\alpha}\overline{W'(z)}$ these "zero modes" are parametrised by only *one* complex constant, ζ. In this case the $\bar{\psi}_+\bar{\psi}_-$ term in eq. (3.15a) is zero, so that

$$z = z_{\mathrm{cl}}(\sigma), \qquad \psi_+ = -e^{i\alpha}\psi_- = (2\partial_\sigma z)\zeta \tag{3.22}$$

is an exact solution of eqs. (3.15a, b) for arbitrary complex (anticommuting) constant ζ if $z(\sigma)$ is a solution of eq. (3.18). Since the soliton configuration $z(\sigma)$ breaks spatial translation invariance, $z(\sigma + a)$ is also another solution for arbitrary (real) constant a. For fixed boundary conditions, therefore, solutions of eqs. (3.15a, b) that saturate the bound $M \geqslant |T|$ are parametrised by the coordinates of a superspace of real dimension $(1|2)$, rather than $(1|4)$. This is taken into account in the superparticle (effective) action by virtue of its fermionic gauge invariance for $M = |T|$.

4. Intersecting domain walls in the WZ model

In the $d = 4$ WZ model the domain walls separate regions which have different vacuum configurations, each vacuum being associated with a critical point of the superpotential $W(z)$. For many superpotentials these domain walls can have stable stringlike intersections, providing explicit examples of intersecting supersymmetric extended objects. In this section we analyse the WZ model with a general quartic superpotential,

$$W = z^4 - \alpha z^3 - \beta z^2 - \gamma z\,, \tag{4.1}$$

to find the conditions under which stable intersections can occur.

Consider first a model with the superpotential

$$W = z^4 - 4z\,. \tag{4.2}$$

This superpotential has three symmetrically placed critical points at $z_1 = e^{2\pi i/3}$, $z_2 = e^{-2\pi i/3}$ and $z_3 = 1$. To each pair of critical points (z_a, z_b) is associated the topological charge

$$T_{ab} = 2\,e^{i\arg(W(z_b) - W(z_a))}\left|W(z_b) - W(z_a)\right|. \tag{4.3}$$

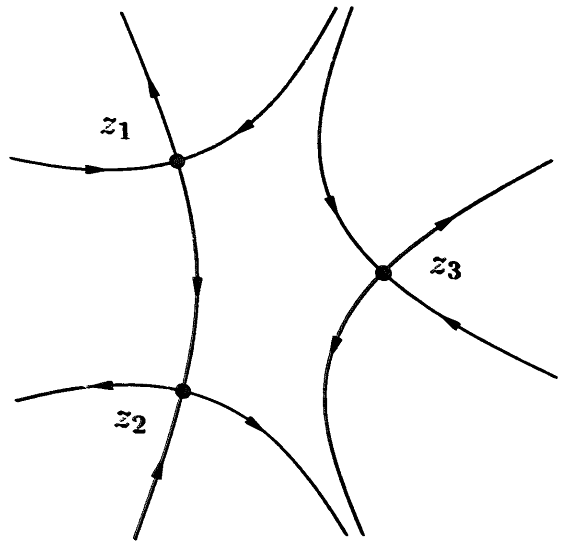

Fig. 1. Phase flow of the soliton equation (3.18) for a superpotential (4.2) with three symmetrically placed critical points, $\alpha = \pi/2$.

Upon dimensional reduction the domain walls become the solitons of the $d=2$ WZ model described in sect. 3, so to find whether there exists a domain wall interpolating between the vacuum configurations $z=z_1$ and $z=z_2$ we have to determine whether the first-order differential equation (3.18), with α equal to $\arg T_{12}$, admits a solution that connects z_1 to z_2. We are therefore interested in the solutions of

$$\partial_\sigma z = 4\,e^{i\pi/2}(\bar{z}^3 - 1)\,, \tag{4.4}$$

which are sketched in fig. 1. We see that there is indeed a solution with the required boundary conditions. For the superpotential in hand, symmetry considerations clearly imply that there exist soliton solutions connecting any pair of the three critical points once this is established for a particular pair. To see this explicitly we need not actually redraw fig. 1 for each of the other two values of α. We have merely to note that in taking $\alpha \to \alpha + \psi$, the flow lines of (4.4) are rotated through an angle ψ at each point of the diagram. The rotated flow lines are therefore curves which cut the original flow lines at constant angle ψ. By continuity, there exists a value of ψ for which the rotated flow line leaving $z=z_1$ will reach $z=z_3$. Because of the symmetry we know that the value of ψ for which this happens is $2\pi i/3$. There are three possible soliton–antisoliton pairs with topological charges of equal magnitude but relative phases (± 1, $\pm e^{2\pi i/3}$, $\pm e^{-2\pi i/3}$). Thus any two solitons of different types may annihilate to leave an anti-soliton of the third type, and therefore in the $d=4$ WZ model with this superpotential there are three domain walls which can intersect.

It might be suspected that a superpotential with three critical points would always allow three domain walls, but one of the domain walls can become unstable as the parameters of the non-leading terms in the superpotential change. For example, the tension M_3 of a domain wall, initially satisfying $M_3 \leqslant M_1 + M_2$, where M_1 and M_2 are the tensions of the two other walls, may increase as the superpotential is changed until $M_3 = M_1 + M_2$. At this point the domain wall becomes unstable against decay into the other two and will therefore not appear as an *exact* solution of the first order equation $\partial_\sigma z = e^{i\alpha}\overline{W'(z)}$ joining two critical points of W (for any value of α).

To see how a domain wall solution can vanish in this way, let us consider the influence of a critical point at $z = z_3$ on a pair of nearby critical points, at $z = z_1$ and $z = z_2$, as the superpotential changes. We can choose $\alpha = \arg T_{12}$, so that if there is a soliton solution interpolating between $z = z_1$ and $z = z_2$ it will appear in the phase portrait of $\partial_\sigma z = e^{i\alpha}\overline{W'(z)}$ as a separatrix between these points. Given a phase portrait for which such a separatrix exists, the way in which it may disappear is illustrated in fig. 2. The potential in fig. 2a allows three soliton solutions (two of them do not appear in the figure as they have other values of α), whereas the potentials in figs. 2b, c allow only two solitons. At the crossover point, illustrated by

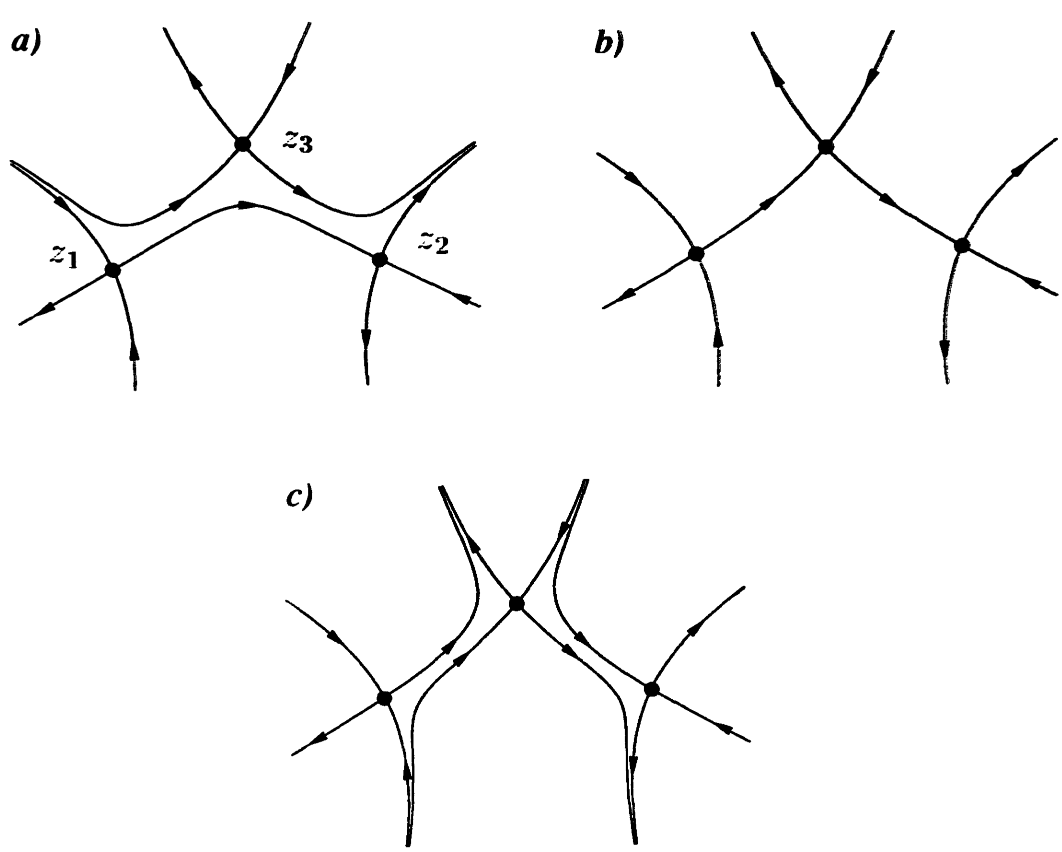

Fig. 2. When the parameters of the potential are changed the domain-wall solution vanishes.

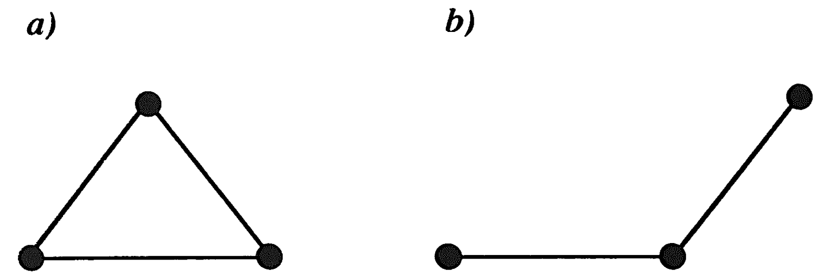

Fig. 3. Possible "networks" of domain-wall solutions connecting the three vacua of a quartic superpotential.

fig. 2b, z_1 and z_2 are both connected to z_3 by separatrices; the corresponding topological charges T_{13} and T_{32} therefore have the same phase. Thus, allowing for all values of α, the three distinct vacua are connected by a "network" of possible domain-wall solutions in one of two ways, either as in fig. 3a (corresponding to fig. 2a) or as in fig. 3b (corresponding to figs. 2b, c).

As an illustration we now turn to the general quartic superpotential. By rescaling and shifting z by complex constants we can always arrange (provided that not all three critical points are degenerate) that W of (4.1) take the form

$$W = z^4 - \tfrac{4}{3}\mu z^3 - 2z^2 + 4\mu z\,, \tag{4.5}$$

up to multiplication by a complex number which can be absorbed by a rescaling of σ and a shift of α. For example, the symmetric potential (4.2) is equivalent to the potential (4.5) with $\mu = \pm i\sqrt{3}$. Thus the physics depends on the single complex parameter $\mu = \mu_1 + i\mu_2$. The three critical points are given by

$$z_1 = -1, \qquad z_2 = 1, \qquad z_3 = \mu. \tag{4.6}$$

We have seen previously that at any boundary separating regions in which the three critical points have different connectivities two topological charges must have the same phase. This happens if either $\mu_2 = 0$ or

$$3\mu_1^4 + 2\mu_1^2\mu_2^2 - \mu_2^4 - 6\mu_1^2 - 6\mu_2^2 + 3 = 0. \tag{4.7}$$

There is no change in the stability of the domain walls in crossing the real μ-axis but there is in crossing one of the boundaries defined by (4.7). These boundaries are shown in fig. 4.

For a generic polynomial superpotential, of order $k+1$, there will be k isolated critical points in the z-plane connected by some network of solutions to the equation $\partial_\sigma z = e^{i\alpha}\overline{W'(z)}$, for appropriate values of α. All the critical points are saddle points from which it follows that the integral curves of this equation, *for given* α, cannot form closed loops. Hence there is at most one separatrix joining

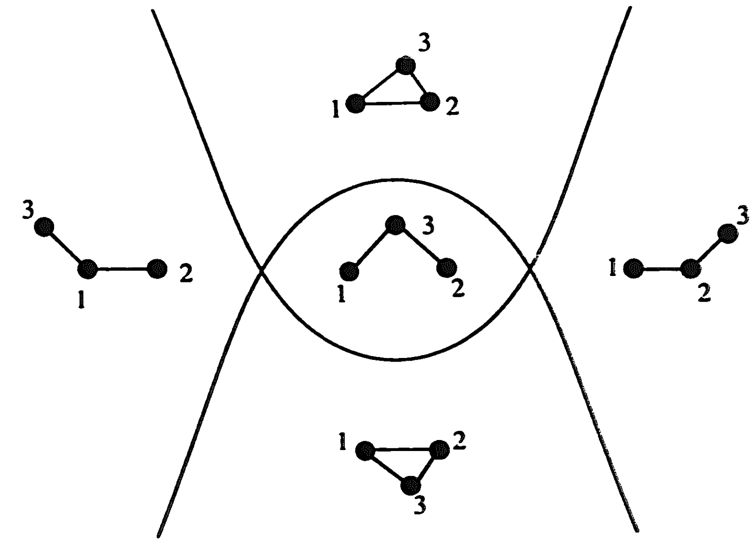

Fig. 4. The connectivity of the three critical points in different regions of the parameter space of a quartic superpotential.

any pair of critical points and if there is such a separatrix it will occur for a unique value of α, modulo π. There will therefore be at most one solution linking any pair of critical points (which can be traversed in opposite direction by taking $\alpha \to \alpha + \pi$). As the parameters of W are continuously changed the connectivity of this network may change. Domain-wall intersections will be possible whenever the network of connected vacua has loops, the domain walls that meet at the intersection corresponding to the links of the loop. For higher-order superpotentials it will be possible to have intersections which involve more than three domain walls. For example a generic quintic superpotential will have four isolated critical points. There are in principle six possible ways in which these critical points might be connected by domain-wall solutions, as shown in fig. 5, although we do not know whether all possibilities are realised for some region of the (four-dimensional) parameter space of physically distinct quintic superpotentials.

If the connectivity is as in figs. 5a or 5b no stable domain-wall intersections are possible. If it is as in figs. 5c, 5d or 5e, intersections of three domain walls are possible; for figs. 5d or 5e intersections of four domain walls are also possible but they can be viewed as superpositions of intersections of three walls. If the connectivity is as in fig. 5f an intersection of three walls is not possible but one of four is.

The polynomial superpotentials we have considered so far are all perturbations of the degenerate superpotential $W = z^{k+1}$. Under a generic perturbation by lower powers of z the degenerate critical point of this superpotential at $z = 0$ is resolved into k isolated non-degenerate critical points. If we had restricted discussion to interactions of the scalar field z that are renormalisable in $d = 4$, then $k = 2$ would have been the only possibility, leading to at most one type of domain wall,

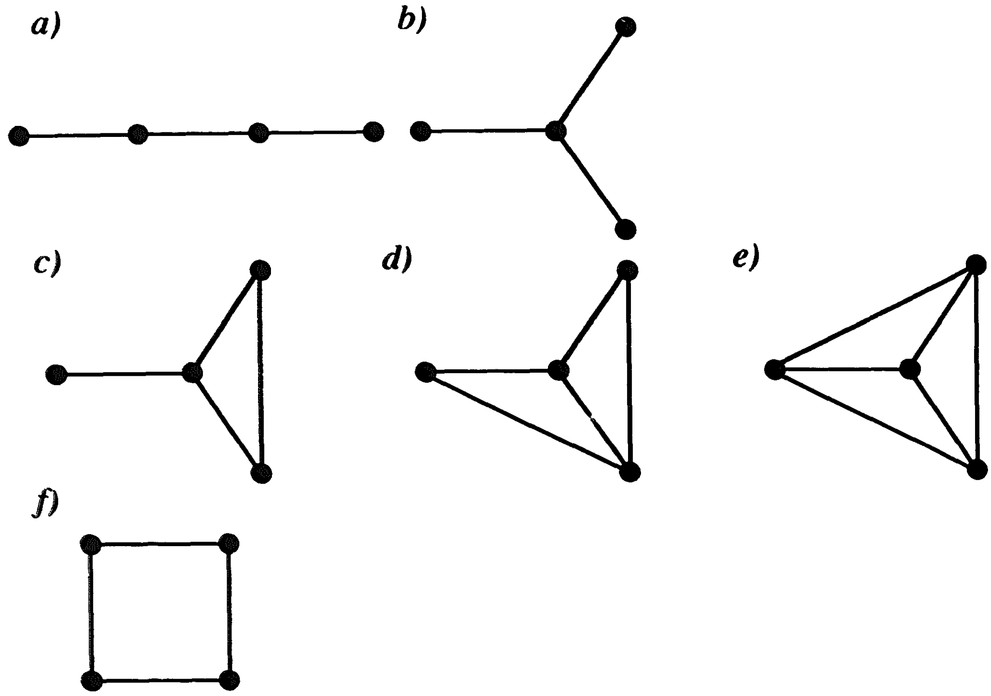

Fig. 5. Possible networks of domain-wall solutions for a quintic superpotential.

for which the topological charge can be taken to be real and for which stable domain wall intersections therefore cannot form. To find renormalisable potentials for which stable domain wall intersections can form it is therefore necessary to consider superpotentials that depend on at least two complex variables. For a superpotential that depends on just two complex variables z_1 and z_2 it remains true that the critical points are isolated. In this case the possible degenerate critical points belong to one of the A, D or E series [16]. The restriction to field theories that are renormalisable in $d=4$ now yields one further possibility: $W=z_1^2z_2+z_2^3$. The critical point at $z_1=z_2=0$ is of type D and multiplicity 4, i.e. D_4. Under a generic perturbation by the monomials (z_1, z_1^2, z_2) the degenerate critical point is resolved into four non-degenerate critical points, i.e. a four-well potential. In this case there will certainly be at least three types of domain wall and it seems likely that stable intersections will be able to occur for some regions of the parameter space formed by the coefficients of (z_1, z_1^2, z_2). If there are more than two scalar fields then non-isolated critical points will be generic.

5. Conclusions

Motivated by Strominger's recent suggestion that the dynamics of stringlike intersections of 5-brane solutions of $d=10$ supergravity/Yang–Mills should be governed by the action of the Green–Schwarz superstring, we have investigated the general conditions under which extended object solutions of supersymmetric field theories may be expected to form stringlike intersections.

Firstly we have noted that there are two other cases, i.e. in addition to the one suggested by Strominger, in which this might be expected to happen. Curiously, they are precisely the maximally-extended objects of the $\mathbb{R}, \mathbb{C}, \mathbb{H}$ families in the classification of supersymmetric extended objects, suggesting that the notion of intersections provides the unifying link between these families.

Secondly we have shown that the conservation of the antisymmetric tensor topological charge carried by extended objects of supersymmetric field theories provides a severe constraint on the energetics of their intersections, sufficient to prevent stable intersections from forming unless there is an internal degree of freedom allowed to the topological charge. An example for which there is such a freedom is that of domain walls in the $d = 4$ Wess–Zumino model with holomorphic superpotential W. In that case the topological charge is complex and a difference in the phase of this charge for intersecting domain walls is crucial to the stability of the intersection. We have presented a detailed analysis for a quartic superpotential, and shown that for certain regions of parameter space domain walls will indeed intersect on strings.

In contrast, there is no internal degree of freedom allowed to the 5-index topological charge of a $d = 10$ 5-brane, a fact that is related to the absence of a non-trivial (continuous) automorphism group of the $(1,0)$ $d = 10$ supersymmetry algebra. It follows from our general arguments that intersections of 5-branes cannot be more than marginally stable. At this point we should emphasise that these arguments strictly apply to flat space $d = 10$ field theories because it is only in this case that the supersymmetry charge is locally well defined. Hence while the argument applies to the 5-brane solitons of the pure $d = 10$ super YM theory , it might fail for the analogous heterotic 5-brane solution of ref. [3], which involves gravitational fields. Another escape from our conclusion would be to consider 5-brane solutions that wrap around a compactified dimension and that have a non-zero momentum in this direction. This could bring other charges into play.

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Note added in proof

That the central charge of solitons in the $d = 2$ $(2,2)$-supersymmetric Wess–Zumino model is complex has been noted previously by Fendley et al. [17]. These authors have shown that the model with superpotential $W(z) = z^{k+1} - z$ for integer k is integrable. In this case, therefore, the "solitons" are actually solitons in the strict sense of the word, as well as in the loose sense of "particle-like solutions" used here. We thank D. Olive and G. Sierra for bringing this paper to our attention.

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