Physics Letters B 295 (1992) 225–232
North-Holland

PHYSICS LETTERS B

More on Q-kinks: a (1 + 1)-dimensional analogue of dyons

E.R.C. Abraham and P.K. Townsend

DAMTP, University of Cambridge, Cambridge CB3 9EW, UK

Received 24 August 1992

Time-dependent solutions, called Q-kinks, of certain (1 + 1)-dimensional sigma-models with scalar potentials are shown to have many properties in common with BPS dyons of (3 + 1)-dimensional Yang–Mills/Higgs theory. In particular, sigma-model analogues of the phenomena of fractionally charged dyons and of charge-exchange between dyons and axion domain walls are described. In 2 + 1 dimensions Q-kinks become Q-strings. We show that the recently discovered Q-lumps of the (2 + 1)-dimensional model can be considered to be stable loops of Q-string.

1. Dyons and Q-kinks

Dyons are solutions of Yang–Mills/Higgs (YM/H) theories that carry both electric charge, Q_e, and magnetic charge, Q_m [1]. Exact solutions can be found [2,3] in the BPS limit, with an energy $\mathcal{E}$ that saturates the Bogomolnyi bound

$$\mathcal{E} \geqslant \sqrt{Q_e^2 + Q_m^2} \, . \tag{1.1}$$

These solutions obey *first-order* (Bogomolnyi) equations. For the BPS monopole, with $Q_e = 0$, these equations can be interpreted as (anti)self-dual equations for a YM four-vector potential, A, in a locally euclidean space of topology $\mathcal{R}^3 \times S^1$, if the Higgs field is identified as the fourth component of A [4,5]. The electric charge of a dyon can then be interpreted as the component of the momentum around the S^1 factor, so a dyon is a monopole that has been boosted in the "extra" dimension [6]. A feature of this Kaluza–Klein (KK) interpretation is that the compactness of the extra dimension provides an alternative explanation of the quantization of electric charge in the presence of a monopole.

It is well known that (1 + 1)-dimensional sigma-models have many properties in common with (3 + 1)-dimensional YM theory. An additional aspect of this analogy is that a class of sigma-models with a scalar field potential have (in general time-dependent) solitonic solutions, called "Q-kinks", with similar properties to those of the BPS dyons of YM/H theory [7]. Let $\{\phi^I\}$ be the coordinates of a Kähler target space, $\mathcal{M}$, with metric $g_{IJ}(\phi)$, and let $k = k^I(\phi)\partial_I$ be a holomorphic Killing vector. Let $x^m = (t, x)$ be cartesian coordinates for (1 + 1)-dimensional Minkowski spacetime, M_2. Given a map $f\colon M_2 \to \mathcal{M}$ the sigma-model fields $\phi^I(t,x)$ are given by $f\colon (t,x) \mapsto \phi^I(t,x)$. The action with Q-kink solutions of its Euler–Lagrange equations is given by

$$S = \int d^2x \, \tfrac{1}{2}(\partial^m\phi^I \partial_m\phi^J - \mu^2 k^I k^J) g_{IJ} \, , \tag{1.2}$$

where μ is a mass parameter. Denoting derivatives with respect to t and x by, respectively, an overdot and a prime, we can write the energy of any configuration $\phi^I(x)$, for fixed time t, as

$$\mathcal{E}[\phi] = \int_{-\infty}^{\infty} dx \, \tfrac{1}{2}(\dot\phi^I \dot\phi^J + \phi'^I \phi'^J + \mu^2 k^I k^J) g_{IJ} \, . \tag{1.3}$$

0370-2693/92/$ 05.00

Since $\mathcal{M}$ is Kähler it has a complex structure $J^I{}_J$ and a corresponding closed Kähler two-form Ω with components $\Omega_{IJ} = g_{IK} J^K{}_J$. We may make use of this fact to rewrite $\mathcal{E}$ in the form

$$\mathcal{E} = \int_{-\infty}^{\infty} \mathrm{d}x \{ \tfrac{1}{2}(\dot{\phi}^I - \mu v k^I)(\dot{\phi}^J - \mu v k^J) g_{IJ} + \tfrac{1}{2}[\phi'^I - \mu(\sqrt{1-v^2}) J^I{}_K k^K][\phi'^J - \mu(\sqrt{1-v^2}) J^J{}_L k^L] g_{IJ} \}$$
$$+ v Q_e + (\sqrt{1-v^2}) Q_m , \tag{1.4}$$

where v is an arbitrary constant, Q_e is the Noether charge

$$Q_e = \mu \int_{-\infty}^{\infty} \mathrm{d}x\, \dot{\phi}^I k^J g_{IJ} \tag{1.5}$$

associated with the symmetry generated by k, and Q_m is the *topological* charge

$$Q_m = \mu \int_{-\infty}^{\infty} \mathrm{d}x\, \phi'^I k^J \Omega_{IJ} . \tag{1.6}$$

Since k is both Killing and holomorphic it follows that $\mathcal{L}_k \Omega = 0$, where $\mathcal{L}_k$ is the Lie derivative with respect to k. This implies, since $\mathrm{d}\Omega = 0$, that $\mathrm{d}(i_k \Omega) = 0$, where $i_k \Omega$ is the one-form obtained from Ω by contraction with k. This one-form induces a closed one-form on "physical" space (parametrized by x) which is precisely the integrand of (1.6). Hence Q_m is topological. Alternatively, we may use the fact that any holomorphic Killing vector can be expressed locally in terms of a Killing potential [8] $U(\phi)$ by

$$k^I = -\Omega^{IJ} \partial_I U , \tag{1.7}$$

which allows Q_m to be re-expressed in the form

$$Q_m = \mu [U(\phi(x))]_{x=-\infty}^{x=+\infty} , \tag{1.8}$$

for which the topological character of Q_m is evident.

In this model the charges Q_e and Q_m play a role similar to the electric and magnetic charges of the (3 + 1)-dimensional YM/H theory, hence the choice of notation. In particular, by choosing the parameter v such that the unit two-vector $(v, \sqrt{1-v^2})$ is parallel to (Q_e, Q_m) we deduce the energy bound

$$\mathcal{E} \geqslant \sqrt{Q_e^2 + Q_m^2} \tag{1.9}$$

in precise analogy to (1.1). Moreover, this bound is saturated by solutions of the first-order equations

$$\dot{\phi}^I = \mu v k^I , \quad \phi'^I = \mu(\sqrt{1-v^2}) J^I{}_J k^J , \tag{1.10}$$

which are the sigma-model equivalent of the Bogomolnyi equations for YM/H dyons. Q-kinks are solutions of (1.10) that interpolate between zeros of the Killing vector k.

The conditions that $\mathcal{M}$ be Kähler and that the Killing vector k be holomorphic are precisely those required for $N = 2$ [more precisely, (2,2)] supersymmetry [8]. This is again analogous to the 3 + 1 YM/H case since the BPS limit is precisely what ensures the existence of an $N = 2$ supersymmetric extension. In ref. [7] we considered the special case for which $\mathcal{M}$ is *hyper-Kähler* and k *triholomorphic*, which are the conditions required for $N = 4$ supersymmetry. In this case there are three topological charges which, together with the Noether charge, constitute the components of a single quaternionic charge. Here we shall concern ourselves with the more general, Kähler, case. Of course, since hyper-Kähler and triholomorphic are special cases of, respectively, Kähler and holomorphic, the specific models of ref. [7], which were based on the Gibbons–Hawking multi-centre four-metrics, suffice to demonstrate the existence of models admitting Q-kinks. However, since we now require weaker conditions it is possible to find a simpler example.

The simplest example is provided by the choice $\mathcal{M} = S^2$. In coordinates (h, φ), where h is the "height" and φ the azimuthal angle, the metric of a two-sphere of unit radius is

$$ds^2 = \frac{1}{1-h^2}\,dh^2 + (1-h^2)\,d\varphi^2 . \tag{1.11}$$

The non-zero components of the complex structure are

$$J^{\varphi}{}_{h} = -\frac{1}{1-h^2}, \quad J^{h}{}_{\varphi} = 1-h^2 \tag{1.12}$$

and the corresponding closed Kähler two-form is $\Omega = dh \wedge d\varphi$. The holomorphic Killing vector is $k = \partial/\partial\varphi$. For this example eqs. (1.10) reduce to

$$\dot{h} = 0 \Rightarrow h = h(x), \quad \dot{\varphi} = \mu v \Rightarrow \varphi = \varphi_0(x) + \mu v t, \tag{1.13}$$

and

$$\varphi' = 0 \Rightarrow \varphi_0 = \mu y_0, \quad h' = \mu(\sqrt{1-v^2})(1-h^2) \Rightarrow h(x) = \tanh \mu(\sqrt{1-v^2})(x-x_0), \tag{1.14}$$

where x_0 and y_0 are constants. To summarize, the Q-kinks of this model are given by

$$\varphi = \mu(y_0 + vt), \quad h = \tanh \mu(\sqrt{1-v^2})(x-x_0). \tag{1.15}$$

The corresponding topological charge is

$$Q_m = 2\mu . \tag{1.16}$$

Observe that the Killing potential in the chosen coordinates is simply $U = h$, and since the solutions (1.15) interpolate between $h = -1$ and $h = 1$ we confirm that $Q_m = 2\mu$. In general, it can be seen from (1.7) that zeros of the Killing vector correspond to critical points of the Killing potential. This is consistent with the fact that the function $U(\varphi, h) = h$ has no critical points because the metric is singular at precisely those points, $h = \pm 1$, at which the Killing vector vanishes. These singularities are of course merely coordinate singularities and can be removed by a change of coordinates. In non-singular coordinates the Killing potential will have the expected two critical points.

The Noether charge corresponding to (1.15) is

$$Q_e = \mu^2 v \int_{-\infty}^{\infty} dx\,(1-h^2) = \frac{v}{\sqrt{1-v^2}}\,Q_m . \tag{1.17}$$

The energy is therefore

$$\mathcal{E} = \frac{1}{\sqrt{1-v^2}}\,Q_m . \tag{1.18}$$

These formulae have an obvious KK interpretation; v and Q_e can be interpreted as, respectively, the velocity and momentum of a particle of rest-mass Q_m in an "extra" dimension. The factor $1/\sqrt{1-v^2}$ in (1.18) is just the expected relativistic time-dilation factor. It is clear from (1.15) that the constant y_0 can be identified as the coordinate of the extra dimension. Since φ is identified with $\varphi + 2\pi$, y_0 is identified with $y_0 + 2\pi/\mu$. We see that the moduli space of the Q-kink solutions (1.15) is $\mathcal{R} \times S^1$.

Although there is no obvious analogue of the Dirac quantization condition for Q-kinks, the fact that φ is an angular variable ensures that, in the quantum theory, the "electric" charge Q_e will be quantized. However, we remind the reader that there is a subtlety to be considered when determining the electric charge quantum of a YM/H type dyon in a CP-violating theory; if the CP violation is due to a non-zero θ-angle, then the dyon has electric charge $e\theta/2\pi$ modulo an integer [9]. Furthermore, if the constant θ-angle of the YM/H theory is replaced by an axion field $\theta(x)$, having a periodic potential $V(\theta)$, then there can be axion domain walls that interpolate between values of θ that differ by 2π. It has recently been shown [10] that a dyon induces a half-integral electric charge on such a wall and that a dyon that passes through it must exchange electric charge with the wall. We discuss a sigma-model analogue of these result in the following section, with the dyon replaced by a Q-kink.

To complete the analogy of Q-kinks to dyons, we recall that the sigma-model analogue of the euclidean four-dimensional (anti)self-duality YM equations are the euclidean two-dimensional sigma-model equations

$$\partial_i\phi^I = \pm\epsilon_{ij}J^I{}_J\partial_j\phi^J \quad (i,j = 1,2), \tag{1.19}$$

the solutions of which can be interpreted as (static) "lumps" of the (2+1)-dimensional sigma-model *with vanishing scalar field potential* which, for convenience, we refer to as the "massless" (as against "massive") sigma-model (see for example ref. [11]). If we now dimensionally reduce to dimension 1+1 by setting

$$\partial_2\phi^I = \mu k^I, \tag{1.20}$$

then (i) the action reduces to the massive sigma-model of (1.2), and (ii) eqs. (1.19) reduce to $\phi'^I = \mu J^I{}_J k^J$, which is just the equation of a Q-kink with vanishing "electric" charge, i.e., the analogue of the YM/H monopole. Thus Q-kinks of (1+1)-dimensional massive sigma-models bear a similar relationship to lumps of (2+1)-dimensional sigma-models as do dyons of (3+1)-dimensional YM/H theories to instantonic solitons of (4+1)-dimensional pure YM theories.

At this point we should point out that one can *also* consider the massive sigma-model in (2+1) dimensions, for which there are "Q-lump" solutions [12,13]. We still have the Q-kink solutions, of course, but these must now be considered as "Q-strings", and the charges Q_e and Q_m as "electric" and "magnetic" charges-per-unit-length. A closed loop of Q-string has vanishing total "magnetic" charge but can have a new topological charge-per-unit-length, not available to Q-kinks, such that the total charge is an integer. We shall show that this charge can stabilize a closed loop of Q-string; the stable configuration is none other than a Q-lump. An interesting point here, that we have not yet investigated, is whether there is an analogue of this phenomenon in (4 + 1)-dimensional YM/Higgs theory, i.e., a stable closed loop of "dyonic-string", as the Q-kink/dyon analogy that we are here propounding would suggest.

2. Fractional charge and axion-kinks

Because of the identification of φ with $\varphi + 2\pi$, the Q-kink solution (1.15) is periodic with period

$$T = \frac{2\pi}{\mu v}. \tag{2.1}$$

The action for a Q-kink configuration evaluated over one period is $S = -4\pi\sqrt{1-v^2}/v$ and hence

$$S + \mathcal{E}T = 4\pi\frac{v}{\sqrt{1-v^2}}. \tag{2.2}$$

According to semi-classical reasoning (see for example ref. [14]) this quantity must be an integral multiple of 2π. Given the expression (1.17) for Q_e and the fact that $Q_m = 2\mu$ this is equivalent to the quantization condition

$$Q_e = \mu \times \text{ integer}, \tag{2.3}$$

as expected from its KK interpretation.

Following Witten's discussion [9] of dyons in CP non-conserving theories we now consider the addition to the action of the CP-violating total derivative term

$$S_\theta = -\frac{\theta}{4\pi}\int_{\mathrm{M}_2} f^*(\Omega)\,, \tag{2.4}$$

where $f^*(\Omega)$ is the pull-back of the target space Kähler two-form Ω. Since we wish to construct an analogy with YM theory we require that $\mathrm{e}^{\mathrm{i}S_\theta}$ be invariant under $\theta \to \theta + 2\pi$, for any field configuration. Since the integral over $f^*(\Omega)$ is the integral of Ω over the image two-cycle $f(\mathrm{M}_2)$ in $\mathcal{M}$, this is equivalent to the requirement that $(1/4\pi)\Omega$ be an integral two-form, i.e., that $\mathcal{M}$ be a Hodge manifold. Note that this property is satisfied by our simple S^2 example because Ω is the area two-form and its integral over the unit circle is 4π. In components, the addition to the action is

$$S_\theta = -\frac{\theta}{4\pi}\int \mathrm{d}^2x\,\dot{\phi}^I\phi'^J\Omega_{IJ} = \frac{\theta}{4\pi}\int \mathrm{d}t\,\mathrm{d}x\,(\dot{\varphi}h' - \varphi'\dot{h})\,, \tag{2.5}$$

where the second line is the expression for the S^2 model, which is the case we shall deal with in the following.

For the Q-kink configurations of (1.15), evaluated over one period, we have

$$S_\theta(Q\text{-kink}) = \theta\,, \tag{2.6}$$

which must now be added to the right-hand side of (2.2). The resulting modified quantization condition is

$$Q_\mathrm{e} = -\frac{\mu\theta}{2\pi} + \mu\times\text{integer}\,. \tag{2.7}$$

Clearly, the mass parameter μ here plays the role of the electric charge unit of the YM/H theory.

Following ref. [9] we can also establish the result (2.7) by canonical reasoning. For this purpose we rewrite the action, now including the θ-term, in the canonical form

$$S = \int \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}x\left[\dot{\phi}^I\pi_I - \frac{1}{2}\left(\pi_I - \frac{\theta}{4\pi}\phi'^K\Omega_{KI}\right)\left(\pi_J - \frac{\theta}{4\pi}\phi'^L\Omega_{LJ}\right)g^{IJ} - \tfrac{1}{2}\phi'^I\phi'^J g_{IJ} - \tfrac{1}{2}\mu^2k^Ik^Jg_{IJ}\right], \tag{2.8}$$

where $\pi_I(t,x)$ are the canonical conjugate fields to ϕ^I; their Euler–Lagrange equations are

$$\pi_I = \dot{\phi}^J g_{JI} + \frac{\theta}{4\pi}\phi'^J\Omega_{JI}\,, \tag{2.9}$$

from which we see that the effect of the total derivative term in the action is to modify the definition of the canonical conjugate fields. This has a significant effect on the Noether charge Q_e because when expressed in canonical variables it acquires a θ-dependence. Specifically,

$$Q_\mathrm{e} = \mu\int_{-\infty}^{\infty}\mathrm{d}x\,k^I\left(\pi_I - \frac{\theta}{4\pi}\phi'^J\Omega_{JI}\right) = \mu\int_{-\infty}^{\infty}\mathrm{d}x\,k^I\pi_I - \frac{\theta}{4\pi}Q_\mathrm{m}\,. \tag{2.10}$$

For simplicity, consider the Q-kink of the S^2 example, for which $k^I\pi_I = \pi_\varphi$ and $Q_\mathrm{m} = 2\mu$. Then, upon canonical quantization, we find the "electric" charge operator

$$\hat{Q}_\mathrm{e} = -\mathrm{i}\mu\partial_\varphi - \frac{\mu\theta}{2\pi}\,, \tag{2.11}$$

where

$$\partial_\varphi \equiv \int_{-\infty}^{\infty} dx \frac{\delta}{\delta\varphi(x)} \tag{2.12}$$

is a functional differential operator with integer eigenvalues, as a consequence of the identification $\varphi \sim \varphi + 2\pi$. We therefore recover the quantization condition of (2.7).

We now promote the hitherto constant parameter θ to a field $\theta(x)$, which we shall call the axion field by analogy with YM theory. We thus extend our considerations to an action of the form

$$S = \int d^2x \left(\tfrac{1}{2}(\partial^m \phi^I \partial_m \phi^J - \mu^2 k^I k^J) g_{IJ} - \frac{1}{4\pi} \theta \dot{\phi}^I \phi'^J \Omega_{IJ} + \tfrac{1}{2} \partial^m \theta \, \partial_m \theta - V(\theta) \right), \tag{2.13}$$

where the potential $V(\theta)$ is assumed to be periodic in θ with period 2π and to have its absolute minima at $\theta = \theta_0$, modulo 2π. Under these circumstances there will be additional kink-like solutions that interpolate between adjacent minima of V; these are the sigma-model analogues of axion domain walls and we shall therefore refer to them as axion-kinks. As we shall see, there is an interesting effect that occurs when a Q-kink passes through an axion-kink. This effect is closely analogous to a similar, recently investigated [10], effect that occurs when a magnetic monopole, or dyon, passes through an axion domain wall.

The canonical form of the action we are now considering is

$$S = \int dt \int_{-\infty}^{\infty} dx \left[\dot{\phi}^I \pi_I + \dot{\theta}\pi_\theta - \frac{1}{2}\left(\pi_I - \frac{\theta}{4\pi}\phi'^K \Omega_{KI}\right)\left(\pi_J - \frac{\theta}{4\pi}\phi'^L \Omega_{LI}\right) g^{IJ} \right.$$

$$\left. - \tfrac{1}{2}\pi_\theta^2 - \tfrac{1}{2}\phi'^I \phi'^J g_{IJ} - \tfrac{1}{2}\mu^2 k^I k^J g_{IJ} - \tfrac{1}{2}(\theta')^2 - V(\theta) \right], \tag{2.14}$$

and the "electric" charge operator is now

$$\hat{Q}_e = -i\mu\partial_\varphi - \frac{\mu}{4\pi} \int_{-\infty}^{\infty} dx \, \theta(x) k^I \phi'^J \Omega_{JI} = -i\mu\partial_\varphi - \frac{\mu}{4\pi} \int_{-\infty}^{\infty} dx \, \theta(x) h'(x), \tag{2.15}$$

where the latter expression is appropriate for the simple S^2 model. Now consider the effect of passing, quasi-statically, the Q-kink through an axion kink that interpolates between $\theta = \theta_0$ and $\theta = \theta_0 + 2\pi$. If the Q-kink is initially in the region where $\theta = \theta_0$ and carries "electric" charge $[n + (1/2\pi)\theta_0]\mu$ then, as we see from (2.15), it will have a charge of one unit less after having passed through to the region in which $\theta = \theta_0 + 2\pi$. In the process it must transfer one unit of electric charge to the axion kink.

To gain a deeper understanding of this process we rewrite Q_e by an integration by parts in the form

$$\hat{Q}_e = -i\mu\partial_\varphi - \frac{\mu}{4\pi} [\theta(x)h(x)]_{x=-\infty}^{x=\infty} + \frac{\mu}{4\pi} \int_{-\infty}^{\infty} dx \, \theta'(x) h(x)$$

$$= -i\mu\partial_\varphi - \frac{\mu\theta_0}{2\pi} - \frac{\mu}{4\pi} [(\theta(x) - \theta_0) h(x)]_{x=-\infty}^{x=\infty} + Q_e^{(\mathrm{ind})}, \tag{2.16}$$

where

$$Q_e^{(\mathrm{ind})} \equiv \frac{\mu}{4\pi} \int_{-\infty}^{\infty} dx \, \theta'(x) h(x). \tag{2.17}$$

Since $Q_e^{(\text{ind})}$ represents a contribution to the total "electric" charge from those regions for which $\theta' \neq 0$, i.e., at the location of the axion-kink, we may identify this quantity as the charge induced on the axion-kink by the presence of the Q-kink. If the Q-kink and axion-kink are well separated then in the region where $\theta' \neq 0$ it will be a good approximation to set $h = \pm 1$, the sign depending on whether the Q-kink is to the left or to the right of the axion-kink. In this case the induced charge is readily calculated to be

$$Q_e^{(\text{ind})} = \pm \tfrac{1}{2}\mu . \tag{2.18}$$

We conclude, in complete analogy with axion domain walls in the presence of a monopole or dyon, that a Q-kink induces a *half-integral* "electric" charge on the axion kink. As the Q-kink passes through the axion kink the induced charge changes from $-\frac{1}{2}\mu$ to $+\frac{1}{2}\mu$, thus accounting for the transfer of one unit of charge to the axion kink.

The phenomenon of solitons with half-integral charge found here is reminiscent of a similar phenomenon in other $(1+1)$-dimensional field theories [15,16], but there appear to be a number of significant differences. Most obviously, the effect described here requires *two* different types of soliton.

3. Q-strings and Q-lumps

We now return to the massive sigma-model of section 1, but for spacetime dimension $d = 2 + 1$. We still have the Q-kink solutions of the $(1+1)$-dimensional theory, but they should now be interpreted as infinite strings, and the charges (Q_e, Q_m) as charges-per-unit-length. For an infinite string the total charge, of either type, would be infinite. A (finite) loop of string, on the other hand, does not correspond to a solution of the $(1+1)$-dimensional theory. However, a Q-kink configuration is a good local approximation to the string cross-section for a large loop, so it makes sense to ask what will be the *total* charges for such a loop. It is not difficult to see that the total topological charge, $\oint \mathrm{d}\ell\, Q_m(\ell)$, of the loop vanishes because of cancelling contributions from antipodal sections of the string. The total "electric" charge can be non-zero, however, and is given by

$$T_e = \oint \mathrm{d}\ell\, Q_e(\ell) = \int \mathrm{d}^2x\, \dot{\phi}^I k_I . \tag{3.1}$$

Since this charge is also carried by the elementary quanta of the theory, a circular loop of string that shrinks to a region of dimensions of its core can annihilate into these quanta, provided that it carries no other conserved charges.

Actually, there *is* another, topological, charge available to a Q-string, although not to a Q-kink. This charge is the integral over space of the pullback of the closed, integral, Kähler two-form $(1/4\pi)\Omega$,

$$T_m = \frac{1}{4\pi}\int_{\text{Space}} f^*(\Omega) = \frac{1}{4\pi}\int_{\text{Space}} \mathrm{d}h \wedge \mathrm{d}\varphi , \tag{3.2}$$

where the second line is valid for the model with target space S^2. This quantity appeared in the previous section as an instanton number, but with the integral over a $(1+1)$-dimensional spacetime instead of a two-dimensional space. It is also well known as the topological charge of a sigma-model lump but, by Derrick's theorem, the usual static sigma-model lump configurations cannot be solutions of the sigma-model with scalar field potential that we are considering here. However, there is a time-dependent solution carrying non-zero T_e charge known as a Q-lump [12,13]. In fact a Q-lump can be considered as a loop of Q-string carrying a non-zero T_m charge, at least for sufficiently large T_e. To see this, consider a piece of Q-string along the y-axis from y_1 to y_2; then $\mathrm{d}h = \mathrm{d}x\, \partial_x h(x)$, while $\mathrm{d}\varphi = \mathrm{d}y\, \partial_y \varphi(y)$, so for this piece

$$\Delta T_m = \frac{1}{4\pi}\left(\int_{x=-\infty}^{x=\infty} \mathrm{d}h(x)\right)\left(\int_{\varphi(y_1)}^{\varphi(y_2)} \mathrm{d}\phi(y)\right) = \frac{1}{2\pi}[\varphi(y_2) - \varphi(y_1)] = \frac{1}{2\pi}\Delta\varphi . \tag{3.3}$$

For a closed loop $\Delta\varphi = 2\pi\nu$ for integer ν, so we have

$$T_m = \nu . \tag{3.4}$$

If we allow a loop with $T_m \neq 0$ to contract quasi-statically it will reach the minimum energy configuration for given values of the charges T_e and T_m. If both are non-zero this configuration is a Q-lump. Of course, this minimum energy configuration will not necessarily have an obvious interpretation as a loop of string but one can show that for sufficiently large T_e the energy density is indeed ring-shaped. There have been various attempts in the past to find stable string loops, usually called vortons [17], in (3 + 1)-dimensional field theories. In effect, we now see that a Q-lump is an example of a stable vorton, albeit of a (2 + 1)-dimensional field theory. We should mention here that there is actually no finite energy Q-lump for $T_m = \pm 1$ but there is, an least for the S^2 model [13], for $|T_m| > 1$.

It is now easy to see what happens when we attempt to pass a Q-lump through an infinite Q-string separating two different vacua. The Q-lump on one side of the string can be viewed as a "bubble" enclosing a region of the vacuum on the other side. When this bubble meets the Q-string it will simply coalesce with the string, giving up its T_e and T_m charge to the string in the process.

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