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Q-kinks

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Certain (4, 4)-supersymmetric (1+1)-dimensional sigma-models admit a scalar potential. We exhibit models of this type for which there are static particle-like solutions stabilized by a three-vector topological charge $\boldsymbol{Q}$. These solutions form a special class of (generally time-dependent) "Q-kink" solutions stabilized by a quaternionic charge $Q=(Q_0, \boldsymbol{Q})$, where Q_0 is a Noether charge. Q-kinks saturate a Bogomol'nyi-type bound and break half the supersymmetry. As solutions of a four-dimensional field theory they constitute examples of type II supermembranes.

Yang–Mills (YM) instantons can be viewed as soliton-like solutions in a $d=(4+1)$-dimensional YM theory, but since these objects have no definite scale, a perturbation away from exact (anti-) self-duality will cause them tend to spread out indefinitely or shrink to a point. by contrast, BPS monopoles of $d=3+1$ YM/Higgs theory, which can be viewed as solutions of the dimensionally-reduced self-duality equations, have a definite scale and so do not share this instability. Similarly, sigma-model instantons can be viewed as "lumps" of a $d=(2+1)$-dimensional sigma-model but, as in the YM case, they have no scale and therefore suffer from the same instability. In this letter we consider $d=1+1$ sigma-models, obtainable by (non-trivial) dimensional reduction from $d=2+1$, which have "kink"-type solutions of a definite scale that can be considered as $(1+1)$-dimensional analogues of BPS monopoles. The action has the form

$$S=\int \mathrm{d}^2x\, \tfrac{1}{2}(\partial^m\phi^I\, \partial_m\phi^J-\mu^2k^Ik^J)g_{IJ}\,, \tag{1}$$

where $\mathrm{d}s^2=g_{IJ}\,\mathrm{d}\phi^I\,\mathrm{d}\phi^J$ is the metric on the target space $\mathscr{M}$ for coordinates ϕ^I, and $k=k^I\partial_I$ is a Killing vector. This action is obtained from the $(2+1)$-dimensional sigma-model without a potential by imposing $\partial_y\phi^I=\mu k^I$ [1–3] [for spacetime coordinates (t, x, y)]. We shall consider here the particularly interesting special case for which the target space metric is *hyper-Kähler* and the Killing vector *tri-holomorphic*. These models have the feature that they admit a maximal, i.e. (4, 4), supersymmetric extension [2], and many of their properties can be deduced from this fact, as we explain at the conclusion of this letter.

A hyper-Kähler manifold has a triplet of closed Kähler two-forms $\boldsymbol{\Omega}$ and an associated triplet of complex structures,

$$\boldsymbol{J}_I{}^J=\boldsymbol{\Omega}_{IK}g^{KJ}\,, \tag{2}$$

obeying the algebra of the quaternions. A tri-holomorphic Killing vector k is a Killing vector for which $\mathscr{L}_k\boldsymbol{\Omega}=0$, where $\mathscr{L}_k$ is the Lie derivative with respect to k. It follows that $\mathrm{d}(i_k\boldsymbol{\Omega})=0$ and hence that

$$\boldsymbol{Q}=\int_{-\infty}^{\infty} \mathrm{d}x\, \partial_1\phi^I\, k^J\boldsymbol{\Omega}_{IJ} \tag{3}$$

is a *topological* three-vector charge. There is also, of course, the Noether charge

$$Q_0=\int_{-\infty}^{\infty} \mathrm{d}x\, \dot\phi^I k^J g_{IJ} \tag{4}$$

associated with invariance of (1) under $\delta\phi^I \propto k^I(\phi)$.

We now write the energy functional for the action (1) as

$$E=\int_{-\infty}^{\infty} \mathrm{d}x\, \tfrac{1}{2}g_{IJ}(\dot\phi^I\dot\phi^J+\partial_1\phi^I\,\partial_1\phi^J+\mu^2k^Ik^J)$$

$$=\int_{-\infty}^{\infty} \mathrm{d}x\{\tfrac{1}{2}g_{IJ}[\partial_1\phi^I-\mu(\boldsymbol{n}\cdot\boldsymbol{J})^I{}_Kk^K]$$

$$\times[\partial_1\phi^J-\mu(\boldsymbol{n}\cdot\boldsymbol{J})^J{}_Lk^L]$$

$$+\tfrac{1}{2}g_{IJ}(\dot\phi^I-\mu n_0k^I)(\dot\phi^J-\mu n_0k^J)$$

$$+\mu(n_0Q_0+\boldsymbol{n}\cdot\boldsymbol{Q})\}\,, \qquad (5)$$

where $(n_0, \boldsymbol{n})$ is a *unit* euclidean four-vector, i.e., $n_0^2+\boldsymbol{n}\cdot\boldsymbol{n}=1$. By choosing $n=(n_0, \boldsymbol{n})$ parallel to $Q=(Q_0, \boldsymbol{Q})$ we obtain from (5) the energy bound

$$E\geqslant\mu|Q|=\mu\sqrt{Q_0^2+\boldsymbol{Q}\cdot\boldsymbol{Q}}\,, \qquad (6)$$

which is saturated by solutions of the first-order equations

$$\dot\phi^I=\mu n_0k^I,\quad \partial_1\phi^I=\mu(\boldsymbol{n}\cdot\boldsymbol{J})^I{}_Jk^J\,. \qquad (7)$$

Conversely, any solution of eq. (7) (which can be viewed as the sigma-model equivalent of the Bogomol'nyi equations for the BPS monopole) has the property that the four-vector Q is *parallel* to the four-vector n. This can be shown by substitution of eq. (7) into (3) and (4) and use of the quaternion algebra satisfied by the three complex structures.

If $\dot\phi^I=0$ we have a static kink solution with $Q_0=0$. Otherwise we have a time-dependent solution which we call a "Q-kink" by analogy with the Q-lumps of the $d=(2+1)$-dimensional sigma-model [4,5] (and because of the relevance of quaternions). As we shall see later, the possible topological charges are determined by the target space metric and the Killing vector k, i.e., by the parameters of the lagrangian. For a solution with a given topological charge $\boldsymbol{Q}$ there is therefore a one-parameter family of solutions with $Q_0=(n_0/|\boldsymbol{n}|)|\boldsymbol{Q}|$ and energy $E=|\boldsymbol{Q}|/\sqrt{1-n_0^2}$. Static solutions with $n_0=0$ have the lowest energy for given $\boldsymbol{Q}$, but solutions with $n_0\neq0$ are nevertheless stable.

To demonstrate the existence of a model of the required type (the conditions for maximal supersymmetry were given in ref. [2] but it was stated there, erroneously, that no non-trivial potential is compatible with them), and of finite energy solutions to (7), we shall concentrate on the special case for which $\mathcal{M}$ has dimension four. The general hyper-Kähler four-metric with tri-holomorphic Killing vector has the form [6]

$$\mathrm{d}s^2=V^{-1}(\mathrm{d}\phi^0+\boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{\phi})^2+V\,\mathrm{d}\boldsymbol{\phi}\cdot\mathrm{d}\boldsymbol{\phi}\,, \qquad (8)$$

with $\nabla\times\boldsymbol{\omega}=\pm\nabla V$ (in the usual notation of euclidean three-vector calculus), and the three closed Kähler two-forms are

$$\boldsymbol{\Omega}=(\mathrm{d}\phi^0+\boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{\phi})\,\mathrm{d}\boldsymbol{\phi}-V\,\mathrm{d}\boldsymbol{\phi}\times\mathrm{d}\boldsymbol{\phi}\,, \qquad (9)$$

where the wedge product of forms is understood. The general form of V for a complete metric is [6]

$$V=\delta+\sum_{r=1}^{n_c}\frac{2M}{|\boldsymbol{\phi}-\boldsymbol{\phi}_r|}\,, \qquad (10)$$

where $M>0$ and $\delta=0$ or 1. The points $\boldsymbol{\phi}_r$, $r=1, 2, ..., n_c$, are called the "centres" of these "multi-centre" metrics. For a complete metric we also require that $0\leqslant\phi^0\leqslant8\pi M$, so that all these spaces are S^1 bundles over some three-dimensional base space with orbits of $\partial/\partial\phi^0$ as the fibres. The tri-holomorphic Killing vector is $\partial/\partial\phi^0$; it vanishes at the centres $\{\boldsymbol{\phi}_r, r=1, 2, ..., n_c\}$ of the metric where the S^1 radius vanishes. It follows that a metric with more than one centre corresponds to the potential term of (1) having more than one isolated zero. Such a model admits Q-kink solutions, which interpolate between the zeros of this potential.

To see this, we observe that for the special case under consideration eq. (7) reduces to

$$\dot\phi^0=\mu n_0,\quad \dot{\boldsymbol{\phi}}=0,\quad \partial_1\phi^0=0,\quad \partial_1\boldsymbol{\phi}=\mu V^{-1}\boldsymbol{n}\,. \qquad (11)$$

Let $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ be any two centres of the metric. There are finite energy solutions to (11) that interpolate between them, with topological charge $\boldsymbol{Q}=\boldsymbol{\phi}_1-\boldsymbol{\phi}_2$ and Noether charge $Q_0=(n_0/|\boldsymbol{n}|)/|\boldsymbol{Q}|$, provided $\boldsymbol{n}$ is chosen parallel to $\boldsymbol{\phi}_1-\boldsymbol{\phi}_2$. For a two-centre metric with $\delta=0$ the solution is easily found to be

$$\phi^0=\varphi+\mu n_0t\,,$$

$$\boldsymbol{\phi}=\tfrac{1}{2}(\boldsymbol{\phi}_1+\boldsymbol{\phi}_2)$$

$$+\tfrac{1}{2}(\boldsymbol{\phi}_1+\boldsymbol{\phi}_2)\tanh\left(\frac{\mu|\boldsymbol{n}|}{4M}(x-x_0)\right), \qquad (12)$$

where x_0 and φ are constants. In addition to this two-parameter family of (degenerate) solutions there is a one-parameter family of Q-kink solutions labelled by n_0 with a width (and energy) that increases with n_0, so that the narrowest one is the static kink solution.

We turn now to the (4, 4)-supersymmetric extension of our action. A $4k$-dimensional quaternionic-Kähler manifold, $\mathcal{M}$, is one for which the holonomy of a torsion-free connection is contained in Sp(1) $\times$Sp(k). Let f_I^{ia} be a vielbein for $\mathcal{M}$ with $i=1, 2$ a Sp(1)$\cong$SU(2) index and $a=1, 2, ..., 2k$ a Sp(k) index. A hyper-Kähler manifold may now be defined as a quaternionic-Kähler one for which the holonomy is further restricted to lie in Sp(k). The only non-vanishing component of the spin-connection in this case is $\omega_{I\,ia}{}^{jb}=\delta^j_i\omega_{I\,a}{}^b$, where $\omega_{I\,ab}:=\omega_{I\,a}{}^c\Omega_{cb}$ (Ω_{ab} being the antisymmetric Sp(k) invariant tensor) is symmetric in ab. The only non-vanishing component of the Riemann tensor, defined by $R_{IJ\,ia}{}^{jb}:=[\partial_I\omega_{J\,ia}{}^{jb}+\omega_{I\,ia}{}^{kc}\omega_{J\,kc}{}^{jb}-(I\leftrightarrow J)]$, then takes the form

$$R_{ia\,jb\,kc\,ld}=R_{abcd}\varepsilon_{ij}\epsilon_{kl}\,, \tag{13}$$

where R_{abcd} is totally symmetric. The three-vector-valued antisymmetric tensor $\boldsymbol{\Omega}_{IJ}$ may now be written as the SU(2) triplet

$$\Omega^{ij}_{IJ}=2\mathrm{i}f_I{}^{(i}{}_a f_J{}^{j)a}\,. \tag{14}$$

Furthermore, the condition that the Killing vector be tri-holomorphic may be written as [3]

$$k^{(i}{}_{[a;}{}^{j)}{}_{b]}=0\,, \tag{15}$$

which means that $k_{i(a;}{}^i{}_{b)}$ is the only non-zero part of $k_{I;J}$.

With these ingredients we may now write down the (4, 4)-supersymmetric action

$$\begin{aligned}S=\int \mathrm{d}^2x\,[&\tfrac{1}{2}(\partial_=\phi^I\,\partial_{+\!\!+}\phi^J-\mu^2k^Ik^J)g_{IJ}\\&-\tfrac{1}{2}\mathrm{i}\lambda^{ia}_{+}\,\nabla_=\lambda_{+ia}-\tfrac{1}{2}\lambda^{ia}_{-}\,\nabla_{+\!\!+}\lambda_{-ia}\\&+\tfrac{1}{2}\mathrm{i}\mu(\lambda^{ia}_{+}\lambda_{-i}{}^b)k_{ja;}{}^j{}_b\\&-\tfrac{1}{4}R_{abcd}(\lambda^{ia}_{+}\lambda_{+i}{}^b)(\lambda^{jc}_{-}\lambda_{-j}{}^d)]\,,\end{aligned} \tag{16}$$

where $\partial_{+\!\!+}=\partial_0+\partial_1$ and $\partial_==\partial_0-\partial_1$, $\nabla_{+\!\!+}\lambda_{-ia}=\partial_{+\!\!+}\lambda_{-ia}+\partial_{+\!\!+}\phi^I\omega_{Ia}{}^b\lambda_{-ib}$ and similarly for $\nabla_=\lambda_{+ia}$, and the number of plus and minus signs indicates the Lorentz "charge". The supersymmetry transformation laws are

$$\begin{aligned}&\delta\phi^I=\mathrm{i}f^I{}_{ia}(\lambda^{ja}_{+}\epsilon_{-j}{}^i+\lambda^{ja}_{-}\epsilon_{+j}{}^i)\,,\\&\delta\lambda^{ia}_{+}=f_I{}^{ja}\partial_{+\!\!+}\phi^I\epsilon^i{}_{-j}-\mu k^{ja}\epsilon^i{}_{+j}+\delta\phi^I\omega_{Ib}{}^a\lambda^{ib}_{+}\,,\\&\delta\lambda^{ia}_{-}=f_I{}^{ja}\partial_=\phi^I\epsilon^i{}_{+j}-\mu k^{ja}\epsilon^i{}_{-j}+\delta\phi^I\omega_{Ib}{}^a\lambda^{ib}_{-}\,.\end{aligned} \tag{17}$$

The eight (hermitian) supersymmetry changes are $S^{(0)}_{\pm}$ and $S_{\pm}{}^i{}_j$, where

$$\begin{aligned}S^{(0)}_{\pm}=&\int_{-\infty}^{\infty}\mathrm{d}x\,[(\pi\pm\partial_1\phi)^If_I{}^{ia}\lambda_{\pm ia}\\&-\mu k^{ia}\lambda_{\mp ia}+\mathrm{O}(\lambda^3)]\,,\\S_{\pm}{}^{ij}=&-2\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}x\,[(\pi\pm\partial_1\phi)^If_I{}^{(i}{}_a\lambda_{\pm}{}^{j)a}\\&-\mu k^{(i}{}_a\lambda_{\mp}{}^{j)a}+\mathrm{O}(\lambda^3)]\,,\end{aligned} \tag{18}$$

and where $\pi_I=\dot\phi^Jg_{JI}$ are the variables conjugate to ϕ^I; the $\mathrm{O}(\lambda^3)$ terms will not be needed in what follows. We now define

$$a\cdot S_{\pm}:=a_0S^{(0)}_{\pm}+\tfrac{1}{2}a^i{}_jS_{\pm}{}^j{}_i\,, \tag{19}$$

where (a_0, $a^i{}_j$) are components of a euclidean four-vector a. Introducing a further four-vector b, the algebra satisfied by the charges (18) can be summarized as follows:

$$\begin{aligned}&\{a\cdot S_{\pm},b\cdot S_{\pm}\}=2(a\cdot b)(H\pm P)\,,\\&\{a\cdot S_{+},b\cdot S_{-}\}=-2\mu(a\cdot b)Q_0\\&\quad-\mu Q_i{}^j(a_0b_j{}^i-b_0a_j{}^i+\mathrm{i}a_j{}^kb_k{}^i)\,,\end{aligned} \tag{20}$$

where H is the hamiltonian, P the total momentum, $Q_i{}^j=(\boldsymbol{\sigma}\cdot\boldsymbol{Q})_i{}^j$, and $a\cdot b=a_0b_0+\frac{1}{2}a^i{}_jb^j{}_i$. In arriving at this result we have used the canonical (anti) commutation relations that follow from (16). Observe that not only do the topological charges appear as central charges in the algebra, thereby generalizing the result of ref. [7] to $N=4$ supersymmetry, but so also does the Noether charge Q_0 (as noted in the Kähler case in ref. [2]). This is nor surprising from the perspective of this paper because all four charges can be viewed as components of a single quaternionic charge.

A feature of the supersymmetry algebra (20) is that it implies the energy bound (6). To see this, consider the particular hermitian supersymmetry charge

$$S=\tfrac{1}{2}(a\cdot S_{+}+\bar a\cdot S_{-})\,, \tag{21}$$

where a is now a *unit* four-vector and $\bar a$ is the four-vector with components (a_0, $-a^i{}_j$) (i.e., the quaternion conjugate of a). From (20) we then find that

$$\{S,S\}=H-\mu(n_0Q_0+\tfrac{1}{2}n^i{}_jQ^j{}_i)\,, \tag{22}$$

where

$$n = a\cdot\bar{a} = a_0^2 - \tfrac{1}{2}a^i{}_j a^j{}_i , \quad n^i{}_j = -2a_0 a^i{}_j \tag{23}$$

are the components of another unit four-vector n. Since the left-hand side of (22) is positive semi-definite (as an operator in the quantum theory) we have that

$$H \geqslant \mu n\cdot Q , \tag{24}$$

where Q is the four-vector central charge with components $(Q_0, Q^i{}_j)$. The strongest bound is found by choosing n parallel to Q, as we saw before, and the expectation value of (24) then yields (6).

In fact, the bound is saturated by the Q-kink solutions of eq. (7), as we saw earlier. In this case the representations of the (4, 4) supersymmetry algebra with a central charge are shortened (since they may then be viewed as massless representations in $(1+1)+4=1+5$ dimensions). This can also be seen from the fact that a Q-kink breaks only half the supersymmetry. To see this we first rewrite (7) in SU(2) and Lorentz charge notation as

$$\partial_{\pm}\phi^I = \mu f^I{}_{ia} k^{ja} (n_0 \delta_j{}^i + \mathrm{i} n_j{}^i) ,$$
$$\partial_{=}\phi^I = \mu f^I{}_{ia} k^{ja} (n_0 \delta_j{}^i - \mathrm{i} n_j{}^i) . \tag{25}$$

Substituting this into the expressions for $\delta\lambda$ in (17) and requiring that $\delta\lambda=0$ we obtain the following restrictions on the supersymmetry parameters:

$$(n_0 \delta_j{}^i \pm \mathrm{i} n_j{}^i)\epsilon_{\mp}{}^k{}_i = \epsilon_{\pm}{}^k{}_j . \tag{26}$$

These equations are consistent with $\epsilon^i{}_{\pm j} \neq 0$ (since n is a *unit* four-vector); they determine ϵ_+ in terms of ϵ_-, so that there are a total of four independent solutions. the (4, 4) (1+1)-dimensional supersymmetry is therefore broken down to an $N=4$ worldline supersymmetry. The effective action for one Q-kink will be an $N=8$ supersymmetric mechanics with four supersymmetries realized linearly and four non-linearly. the four linearly-realized supersymmetries imply that the worldline "field" content is that of an $N=4$ scalar multiplet with two scalars and four (one-component) spinors. One of the scalars is the Goldstone variable associated with the breaking of translation invariance while the other arises from the breaking of the U(1) symmetry generated by the Noether charge (i.e., from a choice, respectively, of x_0 and φ in (12); there are no Goldstone variables associated with $\boldsymbol{Q}$ because topological charges do not generate symmetries).

In four dimensions the solutions presented in this paper can be interpreted as domain walls of an $N=2$ supersymmetric sigma-model. The effective action for such a well must have $N=2$ worldvolume supersymmetry (in a "physical gauge"; cf. ref. [8] where the $N=1$ action is given). This is therefore an example of a type II supermembrane. If the Goldstone scalar associated with the U(1) symmetry is traded for a vector gauge field (by a duality transformation) the effective action would be similar to the recently proposed type II $d=10$ five-branes [9] and three-branes [10,11]. In our case, however, the $d=4$ type II supermembrane can be viewed more simply as a dimensionally reduced version of the conventional $d=5$ supermembrane.

Finally, we expect that the (4, 4) supersymmetric sigma-models retain their ultraviolet finiteness when a potential is included [2]. It seems likely, by analogy with the WZ models [12], that some choices of the centres of the four-dimensional multi-centre hyper-Kähler metrics will lead to new integrable (1+1)-dimensional field theories.

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