Physics Letters B 278 (1992) 291–296
North-Holland

PHYSICS LETTERS B

Non-linear sigma models and their Q-lump solutions

Edward Abraham
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, UK

Received 6 December 1991; revised manuscript received 12 December 1991

We find the conditions under which the three-dimensional Kähler sigma model with a potential term has non-dissipative but time dependent solutions, called Q-lumps, which saturate a Bogomol'nyi bound. These solutions only exist if the target manifold has a Killing vector field, k^α, with at least one fixed point and if the potential is of the form $V=g_{\alpha\beta}k^\alpha k^\beta$. This potential arises from dimensional reduction and in the linearised theory it is just a mass term. We discuss the elementary properties of the Q-lump solutions and construct explicit examples for the CP^n sigma models.

1. Introduction

Non-linear sigma models have been intensively studied in a variety of different contexts. Although they are not renormalisable in dimensions higher than two they are often used to describe the low energy behaviour of other theories. For an important class of these models, the three-dimensional Kähler sigma models, non-dissipative solutions to the equations of motion can be found. These lump solutions are stabilised by a topological charge and appear as topological defects, in many ways analogous to the monopoles of four-dimensional gauge theories. If, however, the sigma model is regarded as a low energy effective theory it is natural to allow possible potential terms to be included in the lagrangian. It then follows from Derrick's theorem that there can no longer be static lump solutions: the potential will cause them to collapse. It would therefore appear that in the presence of a potential the sigma model has no topological defects. It has, however, recently been found [1] that in the case of a particular sigma model, the CP^1 model, there is a potential for which non-dissipative but *time-dependent* solutions to the equations of motion can be constructed. These solutions, called Q-lumps because of similarities with Coleman's Q-balls [2], have the same form as the pure sigma model lumps but their phase changes with time. In addition to their topological charge they carry a conserved Noether charge which prevents them from collapsing. It is of interest to try to generalise this construction for other models. In this paper we find the conditions under which a sigma model with a potential has Q-lump solutions. We show that for a Kähler sigma model Q-lumps exist only if the target manifold has a continuous isometry with at least one fixed point. For sigma models of this type (which include the CP^n models, for example) the required potential has a particularly simple form. It arises from the dimensional reduction of a pure sigma model in one dimension higher and in the linearised theory it is just a mass term. Because of the naturalness of this potential it can be expected that Q-lumps, or their four-dimensional "Q-string" relatives, will be found in theories of physical significance.

2. The massive sigma model

It is well known that if the target of a sigma model has isometries then these isometries induce an invariance of the sigma model lagrangian. Namely, if ϕ: $\mathbb{R}^3 \to M$ is a map from $(2+1)$-dimensional space-time with the standard flat metric $\eta_{\mu\nu}(+--)$ to a target manifold with riemannian metric $g_{\alpha\beta}$ then the action

$$S=\tfrac{1}{2}\int d^3x\, \eta^{\mu\nu}\, \partial_\mu\phi^\alpha \partial_\nu\phi^\beta g_{\alpha\beta}(\phi) \tag{2.1}$$

is invariant under the infinitesimal transformation

$$\phi^\alpha \to \phi^\alpha + \epsilon k^\alpha , \tag{2.2}$$

where k^α is a Killing vector of the target manifold. The conserved Noether charge induced by this isometry is

$$Q = \int_\Sigma d^2x \, g_{\alpha\beta} \dot{\phi}^\alpha k^\beta , \tag{2.3}$$

where Σ is some spacelike hypersurface. It is this charge that will play the crucial role in the Q-lump construction. We must add a potential term to the lagrangian in such a way that the transformation (2.2) is still a symmetry. The simplest invariant potential is

$$V = g_{\alpha\beta}(\phi) k^\alpha(\phi) k^\beta(\phi) . \tag{2.4}$$

Any function of V would also be invariant under this transformation, and so would be a candidate for a potential. The existence of topological defect solutions is often associated with the existence of a Bogomol'nyi bound. For a bosonic model such a bound invariably indicates that it may be enlarged by the addition of appropriate fermionic terms to form a model with extended supersymmetry. The pure sigma model, for example, may be extended in this way if the target manifold is Kähler or hyper-Kähler, precisely the conditions under which the sigma model has lump solutions. We therefore expect that Q-lumps will be found when the lagrangian with a potential admits such an extended supersymmetric enlargement. It can be shown that the potential (2.4) is the *only* one that can be added to the hyper-Kähler sigma model lagrangian so that it is still consistent with $N=4$ supersymmetry in three spacetime dimensions [3]. There are more general potentials that are consistent with three-dimensional $N=2$ supersymmetry, which is all that we require, but (2.4) is the only one invariant under the symmetry (2.2). The potential (2.4) is therefore a very natural term to add to the sigma model action; we shall see below that it is the unique potential for which Q-lumps can be constructed.

We will therefore look for Q-lump solutions to the equations of motion following from the action

$$S = \tfrac{1}{2} \int d^3x \, g_{\alpha\beta} \, \partial_\mu \phi^\alpha \, \partial^\mu \phi^\beta - m^2 g_{\alpha\beta} k^\alpha k^\beta . \tag{2.5}$$

We assume that the target manifold is Kähler, that is there is defined on the target manifold a covariant constant tensor $J^\alpha{}_\beta$, called a complex structure, which satisfies $J^\alpha{}_\gamma J^\gamma{}_\beta = -\delta^\alpha{}_\beta$. This will be sufficient to ensure the existence of a Bogomol'nyi bound. Our arguments generalise to models with hyper-Kähler targets, manifolds with three independent complex structures, $J^\alpha_{i\beta}$, although we shall not discuss them directly here. We shall refer to (2.5) as the massive sigma model action since when the theory is linearised the potential becomes a mass term.

The action (2.5) has an interesting interpretation as the dimensional reduction of a pure sigma model action in four dimensions, the four-dimensional spacetime being taken to have topology $\mathbb{R}^3 \times S^1$, with the flat metric. Configurations for which the lagrangian is independent of the compact direction must satisfy

$$\phi^\alpha(x + \epsilon \hat{x}_3) = \phi^\alpha(x) + \epsilon m k^\alpha(\phi(x)) , \tag{2.6}$$

where x is an arbitrary spacetime point and $\hat{x}_3$ is the unit vector in the compact direction, which implies that

$$\frac{\partial \phi^\alpha}{\partial x_3} = m k^\alpha . \tag{2.7}$$

Geometrically this means the circles of constant x_0, x_1, x_2 are mapped into the orbits of the Killing vector flow. For this to be possible the Killing vectors must generate a U(1) group of isometries φ_t: $M \to M$. For the field to be single-valued we must require that a translation of $2\pi r \hat{x}_3$, where r is the radius of the compact direction, corresponds to an identity transformation of the isometry group. If the Killing vector flow is parametrised so that $\varphi_{2\pi}$ is equal to the identity then m must be a constant, given by

$$m = n r^{-1} , \tag{2.8}$$

for some integer n. So for configurations satisfying (2.7) we can reduce the pure sigma model action in four dimensions to the three-dimensional massive sigma model action (2.5), up to a factor of $2\pi r$. We can similarly obtain a massive sigma model in any dimension by dimensional reduction of the pure sigma model in one dimension higher. The Noether charge Q can then be interpreted as the component of the momentum in the compact direction. This is analogous to the Kaluza–Klein interpretation of electrically charged monopoles or "dyons" and extreme charged black holes [4]. Both of these can be viewed as dimensional reductions of chargeless objects moving in a five-dimensional spacetime.

This Kaluza–Klein interpretation gives a strong motivation for studying Q-lumps. As many of the candidates for unified theories – most notably string theory – are supersymmetric theories that describe four-dimensional physics via some dimensional reduction procedure, one may expect the low energy dynamics of the scalar fields to be governed by an action of the form (2.5). In four dimensions the scalar fields would then have cosmic "Q-string" defects, corresponding to the Q-lumps of the three-dimensional theory.

3. The Bogomol'nyi equations

The full equations of motion that follow from the massive sigma model action are

$$\partial_\mu \partial^\mu \phi^\alpha + \Gamma^\alpha_{\beta\gamma}\, \partial_\mu \phi^\beta\, \partial^\mu \phi^\gamma + m^2 k_\gamma\, \nabla^\alpha k^\gamma = 0\,, \tag{3.1}$$

where $\Gamma^\alpha_{\beta\gamma}$ and ∇^α are the metric connection and the covariant derivative on the target manifold. These are non-linear second order partial differential equations and so are difficult to solve in general. It is possible, however, to find a special class of solutions which satisfy simpler first order equations, the so-called Bogomol'nyi equations.

The energy of a field configuration is

$$E = \tfrac{1}{2} \int_\Sigma \mathrm{d}^2 x\, \partial_i \phi_\alpha\, \partial_i \phi^\alpha + \dot\phi_\alpha \dot\phi^\alpha + m^2 k_\alpha k^\alpha\,. \tag{3.2}$$

By using the complex structure, $J^\alpha{}_\beta$, this may be rearranged to give

$$E = \int_\Sigma \mathrm{d}^2 x\, \tfrac{1}{4} (\partial_i \phi^\alpha \mp J^\alpha{}_\beta\, \epsilon_{ij}\, \partial_j \phi^\beta)^2 + \tfrac{1}{2} (\dot\phi^\alpha \mp m k^\alpha)^2$$
$$\pm \int_\Sigma \mathrm{d}^2 x\, \omega_{\alpha\beta} \epsilon_{ij}\, \partial_i \phi^\alpha\, \partial_j \phi^\beta \pm m \int_\Sigma \mathrm{d}^2 x\, \dot\phi_\alpha k^\alpha\,, \tag{3.3}$$

where $\omega_{\alpha\beta} \equiv g_{\alpha\gamma} J^\gamma{}_\beta$ are the components of the Kähler form. The third term is just the integral over space of the pull-back of ω,

$$T = \int_\Sigma \phi^* \omega\,, \tag{3.4}$$

and as the Kähler form is closed it is a topological invariant, i.e., it is invariant under smooth deformations of the map ϕ, and must be conserved under evolution of the field by the equations of motion. The fourth term is just the Noether charge (2.3) (multiplied by the coupling constant) and so that too is conserved, it should be noted that no other potential would have led to the conserved charge appearing in this way.

Since the target space metric is riemannian the first two terms are never negative, and so the energy of any configuration must always satisfy the inequality

$$E \geqslant |T| + |mQ|\,. \tag{3.5}$$

This is similar to the Bogomol'nyi inequality for the pure sigma model but it now depends on the value of the charge Q, which is non-topological in origin.

We see, moreover, from (3.3), that the inequality is only saturated by configurations satisfying the first order equations,

$$\partial_i \phi^\alpha \mp J^\alpha{}_\beta\, \epsilon_{ij}\, \partial_j \phi^\beta = 0\,, \quad \dot\phi^\alpha \mp m k^\alpha = 0\,. \tag{3.6}$$

The solutions to these equations are the Q-lumps. It is straightforward to check that they are exact solutions to the full equations of motion. The first equation is just the Bogomol'nyi equation for the pure sigma model, its finite energy solutions are the (anti-)holomorphic maps from the compactified plane into the target manifold. These maps have a natural interpretation. There is a theorem due to Wirtinger (see, for example, ref. [5]) which implies that the only holomorphic submanifolds of a Kähler manifold are those whose induced volume is an absolute minimum among the volumes of all the submanifolds in the same homology class. Moreover, any submanifold whose volume is such an absolute minimum is necessarily holomorphically embedded. This means that the lump solutions are precisely the maps of minimum area.

From the second equation we see that the field must move with or against the Killing vector flow at a rate which depends only on the coupling constant m. Thus if there are Q-lump solutions they will exist for any value of the coupling constant, no matter how small.

It can be seen that the two Bogomol'nyi equations are consistent with one another. The Killing vector flow will preserve the area of any map, relative to the target space metric. It follows from Wirtinger's theorem that this flow must take holomorphic maps into holomorphic maps. Thus a configuration that initially saturates the Bogomol'nyi bound will continue to do so, as of course one would expect on physical grounds.

4. The Q-lump solutions

It appears that from any lump solution of the pure sigma model one can construct Q-lump solutions to the massive sigma model, we have to check, however, that the energy of our Q-lump configurations is finite. The Q-charge of a solution to eq. (3.6) may be written as

$$Q = m \int_{\Sigma} \mathrm{d}^2 x \, g_{\alpha\beta} k^{\alpha} k^{\beta} . \tag{4.1}$$

For finite Q the point at infinity must therefore be mapped to a zero of the potential, corresponding to a fixed point in the Killing vector field. This is not always possible, it may happen that there are no holomorphic spheres embedded in the target manifold on which the Killing vector field has fixed points. For example the manifold $\mathrm{CP}^1 \times \mathbb{C}$, with metric given by the product of the standard metric on CP^1 with the standard flat metric on $\mathbb{C}$, is isometric under translations in the plane, but the corresponding Killing vectors have no fixed points. Thus although a pure sigma model with this target space would have lump solutions it would not be possible to construct Q-lumps from a lagrangian with a potential formed from a Killing vector that generates a translation. While in this case there are other isometries which one could exploit to construct Q-lumps, this need not be so in general.

If the Noether charge is to be finite the fields must also fall off sufficiently fast at infinity; whether they do will depend on the particular model and the particular configuration being considered. Consider, for example, the CP^n model, with the Fubini–Study metric. We have n fields $u_a(z)$, $a = 1, \ldots, n$, and the metric is given by

$$\mathrm{d}s^2 = 4 \frac{\delta_{ab}(1 + \bar{u}_c u_c) - \bar{u}_a u_b}{(1 + \bar{u}_d u_d)^2} \, \mathrm{d}u^a \, \mathrm{d}\bar{u}^b . \tag{4.2}$$

The metric is invariant under unitary transformations of the fields u_a, in particular under the transformation $u_a \to \mathrm{e}^{\mathrm{i}\alpha} u_a$, which rotates each of the fields about the origin. This transformation is generated by the Killing vector $k_a = \mathrm{i} u_a$. If we form the potential from this Killing vector then, defining $z = x_1 + \mathrm{i} x_2$, the Bogomol'nyi equations are simply

$$\partial_{\bar{z}} u_a = 0 , \quad \dot{u}_a = \pm i m u_a , \tag{4.3}$$

and their complex conjugates. These have solutions of the form

$$u_a(z, t) = f_a(z) \, \mathrm{e}^{\pm imt} . \tag{4.4}$$

Since the Killing vector field has a critical point at $u_a = 0$ the functions f_a must go to zero at infinity, for the topological charge to be finite they must be rational [6]. The general Q-lump solutions will then be of the form

$$u_a(z, t) = \sum_{i=1}^{k} \frac{\lambda_{ai}}{z - z_i} \, \mathrm{e}^{\pm imt} , \tag{4.5}$$

where k is the degree of the map. When $|z_i - z_j|$ is sufficiently large this solution represents k component Q-lumps situated at $z = z_i$ with internal parameters λ_{ai}. The Q-charge of these solutions is not, however, necessarily finite; in fact

$$Q \sim \int^{\infty} \frac{\mathrm{d}r}{(\lambda)^2 r + \mathrm{O}(r^3)} , \tag{4.6}$$

where

$$(\lambda)^2 \equiv \sum_a \left(\sum_i \lambda_{ai} \right)^2 . \tag{4.7}$$

So Q will be infinite unless $(\lambda)^2$ vanishes. The parameters λ_{ai} must therefore satisfy the "polygon" equalities,

$$\sum_i \lambda_{ai} = 0 , \tag{4.8}$$

for each a. There is thus not a Q-lump corresponding to every pure sigma model lump. In fact if $k = 1$ there are no Q-lump solutions at all, isolated Q-lumps can only arise as components of multi-lump configurations. The moduli space of Q-lump solutions is correspondingly smaller than in the pure sigma model case. The solutions (4.5) depend on $(n+1)k$ complex parameters which are constrained by the n conditions (4.8). These solutions do not, however, all have the same Noether charge. If we scale the parameters by a factor α we find that $Q(\alpha\lambda_{ai}, \alpha z_i) = |\alpha|^2 Q(\lambda_{ai}, z_i)$. From a solution with a given value of Q we may obtain a solution with any other value of Q by scaling. The moduli space of solutions with a given topological charge which have the same energy is thus one real dimension less than the space of all solutions in that sector. So for any Q the moduli space of Q-lumps has real dimension

$$d=2(n+1)k-2n-1 . \tag{4.9}$$

In the CP^1 case it is easy to visualise the physical significance of these degrees of freedom. When the component Q-lumps are well separated there are $2k$ parameters, z_i, which specify their positions and $2(k-1)$ parameters λ_i, satisfying the polygon equality (4.8), that represent their sizes and phases. Since the positions, z_i, are arbitrary there are no forces between the component Q-lumps: they do not change their energy by moving relative to one another. In the two lump sector the polygon constraint implies that there must be a relative phase difference of π between the two lumps and they must both be of the same size. For a configuration in which the two component Q-lumps each have size λ and are separated by a distance r the Q-charge, by using results from refs. [7,1], may be calculated to be

$$|Q| = 2\pi\lambda^2 p[2K(p)-E(p)] , \tag{4.10}$$

where $p=r/\sqrt{r^2+4\lambda^2}$ and $K(p)$, $E(p)$ are the complete elliptic integrals of the first and second kind respectively. By using the asymptotic expansions of the elliptic integrals we find that for sufficiently large r

$$\lambda(r) \sim \sqrt{\frac{|Q|}{4\pi \ln r}} . \tag{4.11}$$

The size of the component Q-lumps, for a given separation, is therefore fixed by the value of their Q-charge. In topological sectors with more than two component Q-lumps the situation is different. It is now possible for the lumps to change their sizes and phases in a way consistent with the polygon equality. Certainly none of the lumps can go to infinite size: no one lump can have a size greater than the sum of the sizes of the other lumps. However, it is possible for one lump to lose its Q-charge to the others, getting smaller and smaller while the others get closer to a configuration in the sector with one lump less. Thus spiky Q-lumps of arbitrarily small size and large energy density will arise as components of multi-lump configurations.

5. Radiation and stability

It follows from the Bogomol'nyi bound (3.5) that the energy of a Q-lump is the absolute minimum of the energies of all the configurations with the same Q-charge. Since the charge Q is conserved it follows that a configuration which is initially sufficiently close to saturating the Bogomol'nyi bound must always remain close to being a Q-lump solution, so the Q-lumps are stable against classical radiative decay. For the CP^1 model this has been confirmed by numerical simulations [1]. The only instability is a rolling instability: since there is generally a continuous family of Q-lumps which have the same charge and energy an arbitrarily small generic perturbation of a Q-lump will cause the configuration to move further and further from the initial Q-lump solution, but it will always remain near this family.

If, however, a Q-lump is given a perturbation that is large enough to cause it to radiate then it may never return towards a non-singular Q-lump solution. The radiation will carry away Q-charge as well as energy and so after such a perturbation the Q-lump may possible shed all its Q-charge, leaving behind a configuration with a residual non-trivial topological charge that would collapse to a singular spike. It is therefore of interest to study the radiative solutions that carry away energy and charge to infinity. It suffices to study the solutions which are perturbations about the field at infinity $\phi=\phi_c$, ϕ_c being a critical point of the Killing vector field. If we choose $\phi_c=0$ then we are looking for approximate solutions to the equations of motion which are valid for small ϕ. The Killing vectors of $\mathbb{R}^{2n}$ which have a critical point at the origin are, in appropriate coordinates, $k^\alpha = \epsilon^\alpha{}_\beta \phi^\beta$ where $\epsilon^\alpha{}_\beta$ is a complex structure. By using the exponential map we can project these coordinates onto a neighbourhood of the origin of the target manifold so that in this neighbourhood the Killing vector field is just

$$k^\alpha = \epsilon^\alpha{}_\beta \phi^\beta + O(\phi^2) , \qquad (5.1)$$

where $\epsilon^\alpha{}_\beta = J^\alpha{}_\beta|_{\phi=0}$. In these coordinates the connection vanishes at the origin and the equations of motion (3.1) for small ϕ reduce to

$$\partial_\mu \partial^\mu \phi^\alpha + m^2 \phi^\alpha = 0 . \qquad (5.2)$$

This is just the wave equation for a massive scalar field and it has the familiar plane wave solutions. It is equivalent to the first order equation

$$\partial_\mu \phi^\alpha = \pm p_\mu \epsilon^\alpha{}_\beta \phi^\beta , \qquad (5.3)$$

where p_μ is a constant three vector such that $p_\mu p^\mu = m^2$. The Q-charge carried by a radiative solution is therefore

$$Q = \pm p_0 \int d^2x \, \phi_\alpha \phi^\alpha \qquad (5.4)$$

and the total energy is found from (3.2) to be

$$E = p_0 |Q| . \qquad (5.5)$$

As $p_0^2 = (p_i)^2 + m^2$, the radiation will come arbitrarily close to saturating the Bogomol'nyi bound for low frequencies, $\omega = \sqrt{(p_i)^2}$. The charge Q is approximately proportional to the volume of space in which the field is non-zero, so for plane wave radiation the Q-charge is infinite. There are, however, wave packet solutions built from plane waves with a range of momenta. These wave packets are confined to a volume $(\Delta x)^2 \sim 1/(\Delta p)^2$ so their Q-charge is $Q \sim (\phi_0)^2/(\Delta p)^2$, where ϕ_0 is their amplitude. By lowering the frequency of the packet while at the same time increasing its size and adjusting its amplitude one can find radiative solutions that are arbitrarily close to saturating the Bogomol'nyi bound, for any value of Q. Thus although for small perturbations the Q-lumps are stable against radiative decay one cannot immediately rule out the possibility that after a large perturbation the Q-lump will radiate its Q-charge away.

6. Conclusion

We have seen that Q-lumps arise as stable time-dependent solutions to the sigma model with a potential. This potential appears naturally as a result of dimensional reduction for a large class of Kähler (and hyper-Kähler) sigma models and in the linearised theory it can be simply interpreted as a mass term. The Q-lumps saturate a Bogomol'nyi bound and satisfy the first order equations which have a simple interpretation in terms of the geometry of the target manifold.

The scale of the Q-lump solutions is fixed by a conserved Noether charge and so they do not suffer from the rolling scale instabilities of the pure sigma model lumps, although spiky configurations will appear as components of multi-lump solutions.

It would seem that Q-lumps are an interesting addition to the bestiary of topological defects. There are many further issues to explore. We have not discussed at all the low energy dynamics of the Q-lumps, which is known from numerical simulation to be markedly different from that of their pure sigma model relatives [1,8]. The dynamics of the pure sigma model lumps can be understood in terms of the geometry of their moduli space [7,9], it would be satisfying if a similar understanding of the Q-lump dynamics could be obtained. It would also be particularly interesting if "Q-string" solutions could be found in four-dimensional theories of phenomenological interest. The natural origin of the potential term in the massive sigma model suggests that there will be theories of direct physical relevance which contain these solutions.

Acknowledgement

I am particularly grateful to Trevor Samols, Paul Townsend, Gary Gibbons and Robert Leese for enlightening conversations on many of the points discussed in this paper. I also wish to thank the British Council for their financial support.

References

[1] R.A. Leese, Nucl. Phys. B 366 (1991) 283.
[2] S. Coleman, Nucl. Phys. B 262 (1985) 263.
[3] G. Sierra and P.K. Townsend, Nucl. Phys. B 223 (1984) 289.
[4] G. Gibbons and D.L. Wiltshire, Ann. Phys. 167 (1985) 201.
[5] H.B. Lawson, Lectures on minimal submanifolds (Publish or Perish, Berkeley, 1985).
[6] A. D'Adda, M. Lüscher and P. Di Vecchia, Nucl. Phys. B 146 (1978) 63.
[7] R.S. Ward, Phys. Lett. B 158 (1985) 424.
[8] R.A. Leese, Nucl. Phys. B 344 (1990) 33.
[9] P. Ruback, Commun. Math. Phys. 116 (1985) 645.